

D I P L O M A R B E I T

# Market Response Models

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# Market Response Models

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To you.

# Abstract

The purpose of this master thesis is to provide a general up-to-date overview of current approaches and methods in the field of market response models, with the main emphasis on advertising and its dynamics as determining factors.

We will guide the reader through this topic by presenting several distinct approaches, and will on our way use methods from several mathematical fields, including Econometrics, Operations Research, Control Theory and Game Theory, to name the most important. It is this variety of available approaches that we try to subsume into one overall picture in this work, and accordingly to bridge the gap that currently seems to exist between these in marketing literature.

This thesis has a strong theoretical bias, and will only present a few data-based findings, which already have appeared in other publications, along the way. Though the target audience definitely also includes advertising practitioners, who are looking for framework and guidance to better understand, respectively model their markets and its dynamics.

Additionally we devote a complete chapter to the optimality of pulsing policies (respectively to the 'optimal distribution of advertising expenditures over time'), provide an extensive historical overview on this subject in literature, and finally present the latest standings which try to identify the key characteristics within a market which lead to pulsation.

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# Chapter 1

## Introduction

A warning to the reader right away:

*"[...] looking for the relationship between advertising and sales is somewhat worse than looking for a needle in a haystack."* Aaker & Carmen [1] p.68

### 1.1 Definitions and Explanations

**Market response models** try to model market reaction as a function of marketing activities. This includes sales response models as well as market share models.

**Marketing** *is the process of planning and executing the conception, pricing, promotion, and distribution of ideas, goods, and services to create exchanges that satisfy individual and organizational goals.*<sup>1</sup>

Marketing activities therefore include<sup>2</sup>:

- Identifying customer needs

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<sup>1</sup>definition of marketing as approved by the American Marketing Association Board of Directors

<sup>2</sup>cited from <http://www.bsu.edu/marketing/>

- Designing goods and services that meet these needs
- Communicating information about those goods and services to prospective buyers
- Making the goods and services available at times and places that meet customers' needs
- Pricing the goods and services to reflect costs, competition, and customers' ability to buy
- Providing for the necessary service and followup to ensure customer satisfaction after the purchase

Quite commonly these activities are also memorized as the *4 P's*, which are *Product* (e.g. Quality, Diversity), *Place* (i.e. Distribution), *Pricing* and *Promotion*.

**Market reaction** is the overall sum of individual buying decisions made by customers. It can be measured as a quantity measure (e.g. sold units), as a monetary measure (e.g. turnover), or as market share.

The individual reaction functions of the consumers can differ significantly, which leads to the common strategy of *market segmentation*. By splitting the market into several groups according to their (assumed) reaction, and targeting each group separately, it is possible to achieve overall higher revenues.

**Advertising** includes all types of communication intended to bring a product or service to the customer's attention and subsequently persuade them buying it. This therefore not only includes TV commercials and newspaper ads, but also activities like public relation (PR), sponsoring, direct marketing and so forth.

**A model** is a generally simplified mathematical representation of real-world relations. It can be used to provide deeper insight into the ongoing mechanism

(‘Why is something happening?’), to make ‘reliable’ predictions for the future (‘What will happen, if...?’), and/or to determine optimal strategies (‘How should it be done?’).

## 1.2 Concerns & Obstacles of an analytical approach

So what is it, that had Aaker and Carmen come to their pessimistic conclusion in our initial quote? And why is it, that the broad acceptance of market response models has still a rather young history in practice and still has to compete with decisions made by intuition? We will for introductiv reasons highlight two major causes at this point, namely firstly *‘lack of data’* and secondly the *‘inherent complexity’* of market response.

**Available data:** Except for highly aggregated sales data on a yearly basis, it is quite difficult for researchers to gain access to an extensive and reliable data pool. Generally this still requires some type of employment by an accordant organization, and even then relevant key figures (e.g. advertising expenditures of competitors) usually remain unknown and have to be estimated without any underlying data. But not just the data, but even the outcomes of such research tend to remain (for fear of giving away valuable information to competitors) within a company and therefore the progress in this field is overall hampered.

But due to the widespread implementation of computer-aided information systems in organizations the amount of collected data has seen a tremendous increase within the last decades, and it seems that the theoretical research that has been going on for the last fifty years in this area finally finds it application in practice.

Another issue that turns out to be disadvantageous for modelling market response is, that collected data generally just shows little variation. This is partly due to a quite common advertising spending policy that assigns

just a fixed share of current sales to advertising<sup>3</sup>. In order to overcome this problem, experimental testings could be performed within certain market areas of a company. But such a procedure generally encounters serious resistance within a company<sup>4</sup>, since extensive increases as well as decreases of advertising expenditures can lead to enormous costs for the company, whereas the actual benefit of the gained information from these experiments can not be determined reliably in advance.

**Complexity:** There are numerous factors which seem to play a role in the buying decision process for a single customer. These sales drivers include among others the product price, the amount and quality of advertising (TV, billboards, sponsoring,...), product quality, product availability, placement in store, brand image, color of packaging, consumer income, significance of word-of-mouth, and many more that can be possibly included into a market response model. And further, each of these variables can have a differing (possibly asymmetric<sup>5</sup>) effect in relation to the other variables, to its own historic levels, or to the corresponding values of the competitors. This high number of possibilities should make it clear, that quite a bit of experience and understanding of the mechanism in a particular market are required from the model builder to decide on variables and functional forms a-priori.

### 1.3 Benefits & Rewards of an analytical approach

After recognizing the difficulties inherent in modelling market response, the question for the practical value of such research arises.

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<sup>3</sup>The well-known Dorfman-Steiner theorem (see Feichtinger [7] p.314) for example justifies such a policy, if constant price- and advertising-elasticities are assumed.

<sup>4</sup>Ambar Rao provides an in-depth discussion regarding this problematic in [21] chapter 2, whereas he was fortunate enough to be employed by a company who granted him the necessary freedom to design such experiments.

<sup>5</sup>Asymmetric here means that an upward change can yield a different reaction (in size and in shape) than a downward change.

Questions that need to be answered by advertising practitioners are among others: How much money should be spent on marketing/advertising overall? How should this budget be allocated among the market areas, or respectively among media? How to distribute advertising activities over time? What part of my budget should be spend on the production of a campaign and how much on pretesting it?

Furthermore there are issues regarding the optimal marketing mix, i.e. on how to combine price reductions and advertising campaigns in order to achieve the best result. Then there are also concerns of how to react to competition: What should be done if competitors are underpricing? Should this be answered by even lower prices, or compete by taking other marketing measures?

Well, to none of these questions the researcher is likely to find unique universally valid answers, and he will not find them in this thesis either. But what is available, are frameworks on how to model market response, which will guide the decision maker towards an optimal answer and which should reduce/minimize the guesswork that still seems to enjoy great popularity in marketing, even in these days.

## 1.4 Structure of this thesis

This thesis consists (taking this introductory chapter aside) of four main chapters.

We start out by discussing several different shapes of sales response functions in chapter 2, and look at the most common variables, and their possible dynamic interactions.

In chapter 3 we introduce methods of control theory (in particular Pontrjagin's maximum principle) and demonstrate their application in marketing.

In chapter 4 we will make use of this knowledge in order to find answers to the highly interesting question of how to optimally allocate advertising over time.

And finally, chapter 5 deals with game theoretic approaches, which prove useful in order to cope with competitors behavior properly.

# Chapter 2

## Sales Response Functions

One of the primary goals of marketing science is to provide a structural insight of how current and future sales are determined in a market. More precisely, we are interested in the estimation of the sales, resp. market share response function in order to have a better knowledge of future market movements.

Building models generally involves three stages: First the selection of the relevant variables, second the determination of the functional relation between them, and third the estimation of the actual parameters of the model. This chapter will be mainly devoted to the first two tasks. We will be presenting numerous models ranging from simple linear models, over powerful attraction models to highly flexible artificial neural network models. As the models reach higher level of sophistication, generally more complex relations are able to be modelled correctly, but also the more difficult their handling (respectively their estimation) will get.

For an in-depth discussion of the actual estimation of sales response functions through methods of econometrics and time series analysis, the reader is advised to turn to the excellent book 'Market Response Models' by Hanssens, Parsons & Schultz [12].

The dependent variable in such models can either be a quantity measure (e.g. sold units), a monetary measure (e.g. turnover), or a proportion (e.g. market share). Practitioners should be aware of what they want to achieve with their

model before deciding on a particular one. Market share models are generally said to be more robust in respect to external influences (e.g. economic trends, inflation, seasonality). A 20% increase in sales for example is not that significant anymore, if the overall market has doubled during the same time. On the other hand, the number of sold units is the decisive figure for production planning, which should be known as early as possible in order to adjust production accordingly. Note that with a monetary measure problematic correlation between dependent and independent variables might appear, if price is also used as an explanatory variable in the model.

## 2.1 Functional Form

In the following a number of common functional forms together with their characteristics will be discussed<sup>1</sup>. Figure 2.1 should help to provide a general feeling of the actual shapes of such models. Keep in mind that for each of the used variables in the overall model a different model can be used, and that also all kind model combinations are feasible.

### 2.1.1 Linear Model

$$q = \beta_0 + \beta_1 x_1 + \dots + \beta_k x_k$$

Due to its simplicity this model is still commonly used, although it clearly contradicts numerous market characteristics. E.g. linear models assume constant returns to scale, which implies that each additional unit in advertising would lead to an equal incremental change in sales. Furthermore no interaction among the explanatory variables can be explained by such a model. Nevertheless advertising practitioners have a well advanced, powerful set of methods at hand for estimating and testing parameters.

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<sup>1</sup>The classification and notation is taken from Hanssens and Parsons [4] p.413ff.

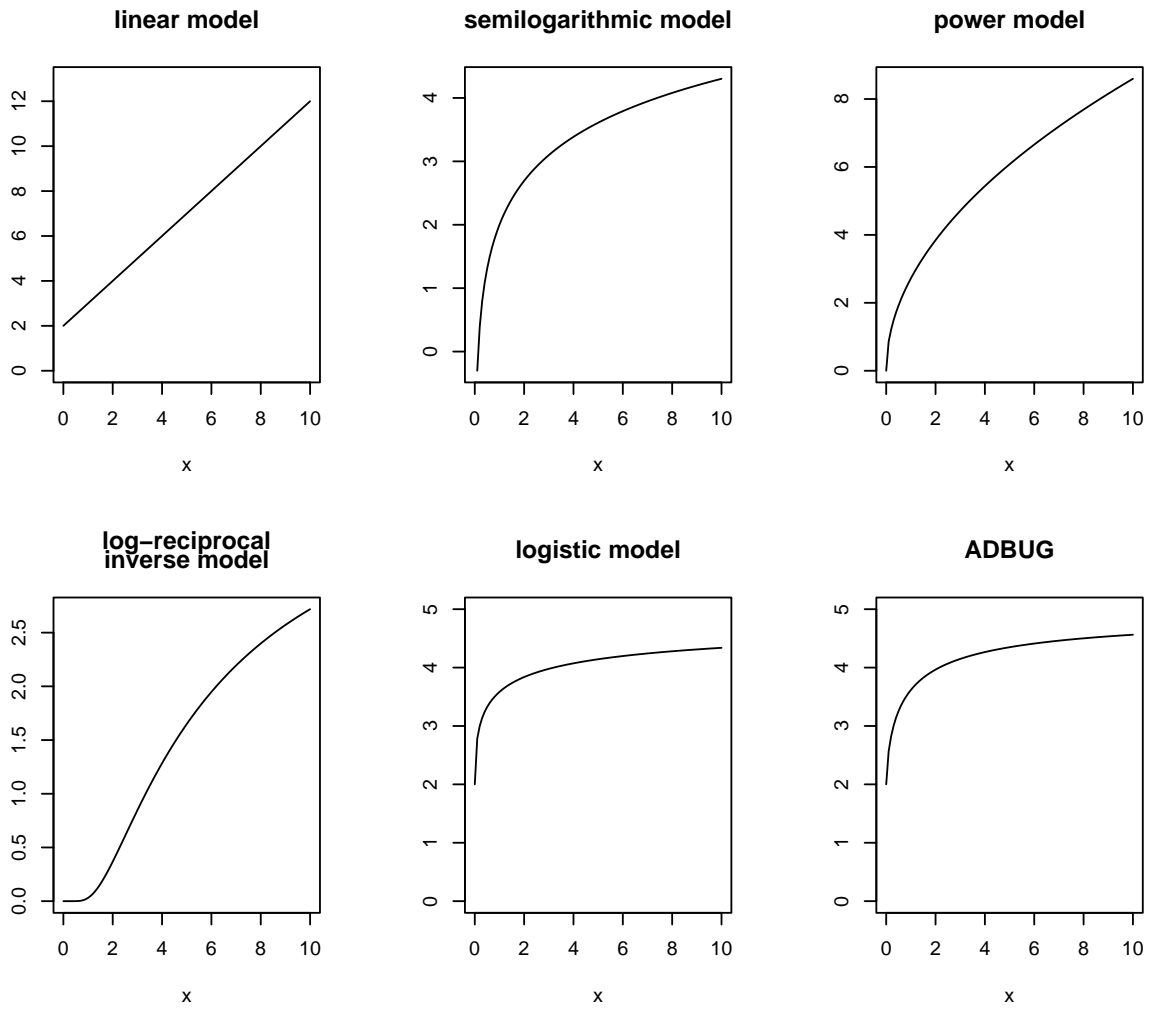


Figure 2.1: exemplary shapes of sales response functions

The reason why linear models are able to show such a (surprisingly) good fit to real data might be that the available observed data generally shows very little variance. I.e. we generally operate in a small subspace of the complete parameter space, so that a linear approximation of the actual functional relation turns out to be sufficiently good in a local context. But advertising practitioners should be cautious with extending a linear model from a local to a global scope and be aware that this might lead to false conclusions (especially when trying to derive optimal policies from such a model) .

### 2.1.2 Multiplicative Model

$$q = e^{\beta_0} x_1^{\beta_1} \cdots x_k^{\beta_k}, 0 < \beta_i < 1 \text{ for } i = 1..k$$

In order to estimate multiplicative models the logarithm can be applied to the equation which yields a linear model.

$$\ln q = \beta_0 + \beta_1 x_1 + \dots + \beta_k x_k, 0 < \beta_i < 1$$

With a multiplicative model it is possible to model *diminishing returns to scale*: A common observation is, that each additional unit of a marketing instrument will subsequently increase sales, but these generated increments will become less and less at higher levels. Basically this transfers to an increasing, strict concave response function.

Another advantage of the multiplicative model is, that the power coefficients  $\beta_i$  can be directly interpreted as the *elasticity* of that particular instrument:

$$\varepsilon_i = \frac{\partial q}{q} / \frac{\partial x_i}{x_i} = \frac{\partial q}{\partial x_i} \frac{x_i}{q} = \beta_i \frac{q}{x_i} \frac{x_i}{q} = \beta_i$$

An obvious downside of the multiplicative model would be that as soon as a single market instrument is not used (i.e. equals 0), the product evaluates to 0, and therefore no sales would occur within such a model. If we have several different marketing instruments in our model, then this is generally a rather unrealistic assumption.

With the following models we will focus on the relation of sales with respect to a single other variable (e.g. advertising), and neglect the interactions between the explanatory variables for now.

### 2.1.3 Semi-Logarithmic Model

$$q = \beta \ln x$$

In this model, which also has a concave shape, a constant *percentage* increase in  $x$  will lead to a constant *absolute* increase in sales. Hermann Simon, for example, used such a relation for his sales response model in ADPULS (see chapter 4.3). A problem of the logarithmic function is its behavior close to zero (where sales would diverge towards minus Infinity), which is commonly tried to be circumvented by adding a constant (e.g. 1) to the marketing effort  $x$ .

### 2.1.4 Modified Exponential Model

$$q = Q^o(1 - e^{-\beta x})$$

It should be clear that regardless of how much effort is put into marketing, that there is a certain upper bound for sales. This maximum sales potential is usually referred to as *saturation level*, and is here denoted with  $Q^o$ . Obviously the modified exponential model is an example for a model which explicitly incorporates such a saturation level:  $\lim_{x \rightarrow \infty} q(x) = Q^o$ . Note, that despite their popularity neither a linear nor a multiplicative model are able to reflect saturation appropriately.

### 2.1.5 Log-Reciprocal Model

$$q = e^{\beta_0 - \beta_1/x}, \beta_0 > 0$$

The models presented so far have been all concave, a property of the sales response function which is not taken for granted by all marketing researchers. There is also some belief that the response function is actual S-shaped, i.e. has a convex and subsequently a concave section. The reasoning behind such a shape is a so-called *threshold effect* takes place, i.e. the phenomena that marketing efforts are not effective until they exceed a certain minimum level.

But it should be noted that there seems to be hardly any empirical evidence<sup>2</sup> for such S-shaped responses. The reason why this issue is so difficult to resolve, is that companies usually operate in the concave part anyways, and therefore just few data exist which could support one or the other hypothesis.<sup>3</sup>

As can be seen from figure 2.1 the log-reciprocal model is able to model such a S-shaped curve<sup>4</sup>.

### 2.1.6 Logistic Model

$$\ln\left(\frac{q - Q_o}{Q^o - q}\right) = \ln \beta_0 + \beta_1 x$$

Similar to the saturation level, we can also incorporate a minimum level (the so-called *base sales*), which we denote with  $Q_o$ . This sales level is obtained when no marketing effort at all is present. The logistic model incorporates base sales, a saturation level, and an S-shaped function simultaneously.

### 2.1.7 ADBUG Model

$$q = Q_o + (Q^o - Q_o) \frac{x^{\beta_2}}{\beta_3 + x^{\beta_2}}$$

---

<sup>2</sup>see Hanssens and Parsons [4] p.438 for a similar statement

<sup>3</sup>This discussion will be of particular relevance for chapter 4, where we will see that S-shaped sales response functions are one of the key factors which can lead to pulsing policies as optimal.

<sup>4</sup>The parameters  $\beta_0 = 1.5$  and  $\beta_1 = 5$  were used to produce the graph.

The logistic model requires information about  $Q_o$  and  $Q^o$  before the actual estimation. A functional form which would also allow these two parameters to be estimated, is the ADBUG model by Little.

### 2.1.8 Quadratic Model

$$q = \beta_0 + \beta_1 x - \beta_2 x^2$$

*Supersaturation* is the phenomena of decreasing sales when marketing efforts are pushed above a certain level. Ambar Rao presents a sales response function with this property in Rao [21] p.20. The quadratic model is another example for models incorporating supersaturation. It is certainly arguable to what extend such an effect might really occur. Since companies usually operate well below such a level, models which do not explicitly incorporate supersaturation usually also prove to be adequate enough for the actual operating range.

### 2.1.9 Transcendental Logarithmic Model

After modelling each marketing effort separately we now turn back again to the interactions among variables. It should be clear that the success of one marketing instrument may very much depend on the simultaneous use (resp. non-use) of others. A price promotion for example is hardly ever performed by companies without a corresponding advertising campaign.

One possibility to incorporate these interactions would be, to have the *parameter* of one marketing effort depend on another marketing effort. Udo Wagner for example models in his paper [26] price elasticity in dependency on advertising. Another, rather general approach, is the transcendental logarithmic model:<sup>5</sup>:

$$\begin{aligned} \ln q = & \beta_0 + \beta_1 \ln x_1 + \beta_2 \ln x_2 + \beta_3 \ln x_3 \\ & + \beta_{12} \ln x_1 \ln x_2 + \beta_{13} \ln x_1 \ln x_3 + \beta_{23} \ln x_2 \ln x_3 \\ & + \beta_{11} (\ln x_1)^2 + \beta_{22} (\ln x_2)^2 + \beta_{33} (\ln x_3)^2 \end{aligned}$$

---

<sup>5</sup>we assume three explanatory variables here

The obvious downside of the newly won flexibility of our model is the high number of parameters which need to be estimated. Therefore it is common practice to apply a-priori restriction on the parameters.

### 2.1.10 Attraction Models (MCI, MNL)

In case that we want to model market shares as the dependent variable, we might be looking for models which have the desirable property of always providing a *logically consistent* solution for given input, i.e. that the resulting market shares range from 0 to 1, and that they sum up to 1. In contrast to the response functions discussed in this chapter so far, attraction models do have this property<sup>6</sup>:

$$m_i = \frac{A_i}{\sum_{j=1}^n A_j}, i = 1..n$$

$$A_i = e^{\alpha_i + \varepsilon_i} \prod_{k=1}^K f_k(X_{ki})^{\beta_{ki}}, i = 1..n$$

$A_i$	the <i>attraction</i> of brand $i$
$X_{ki}$	measure of the use of marketing instrument $k$ for brand $i$
$\beta_{ki}$	parameter of marketing instrument $k$ for brand $i$
$n$	number of brands
$K$	number of marketing instruments
$\alpha_i$	constant parameter for brand $i$
$\varepsilon_i$	residuals

Depending on the function  $f_k$  we can distinguish between several different types of attraction models, which can be combined if necessary. In particular the two most common types are the *MCI* and the *MNL* model:

MCI	<b>m</b> ultiplicative <b>c</b> ompetitive <b>i</b> nteraction	$f_k(X_{ki}) = X_{ki}$
MNL	<b>m</b> ultinomial logit <b>m</b> odel	$f_k(X_{ki}) = e^{X_{ki}}$

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<sup>6</sup>The used notation is taken from Schneider and Tietz [23] p.13.

As can be seen from above, market share is modelled as a function of the usage of all marketing instruments of all competing brands in the market<sup>7</sup>. The more the competitors advertise for example, the bigger the denominator will become, and therefore the less market share can be obtained. But note, that the structure of these models imply, that a change in  $X_{ki}$  has a *symmetric* effect on all other brands. An assumption which sometimes can not be hold, as for example a study on the German chocolate market demonstrated [23] (see in particular page 57 in the referred paper for the detected asymmetries). Another downside is the high number of parameters involved in attraction models, which might be the main reason why, despite their inconsistency, additive or multiplicative market share models are still that popular.

### 2.1.11 Alternative Model Buildings

In the following we will present three "alternative" approaches to obtain the sales response functions: artificial neural networks, non-parametric kernel estimation and structural equation models. The application of all three of them in marketing science has a relatively young history, and therefore the number of published papers is still relatively small.

While reading the following sections, it should be kept in mind, that all of these methods can be and actually are used in combination with other models. For example neural networks could be used for modelling the influence of price within an MNL model, whereas the other explanatory variables are modelled as usual.

#### Artificial Neural Networks

Artificial Neural Networks have become popular due to their flexibility. Loosely speaking, any kind of continuous function can be approximated arbitrarily well

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<sup>7</sup>Be aware, that therefore the estimation of attraction models requires data about competitor's advertising spending, pricing policy, and so on, which might not be available.

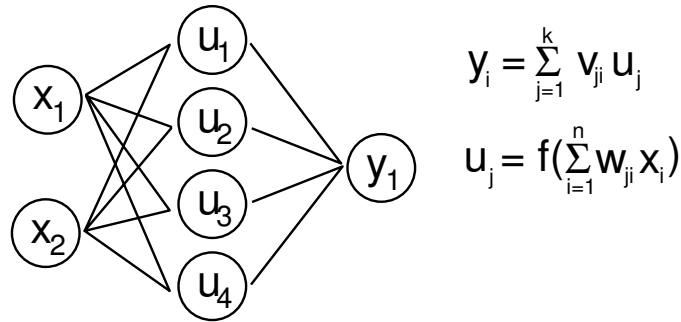


Figure 2.2: artificial neural network

via a single layer perceptron<sup>8</sup>. The model builder does not have to build up the response function guided by his knowledge and assumptions of the market theory, but rather lets the data itself determine the functional shape. Obviously such an approach requires by far more data, and will only be useful if prediction is performed within the range of the available data. Another downside is that the estimated function does not provide any further insight via the estimated parameters, since they allow no particular interpretation.

Figure 2.2 illustrates the general structure of a single layer perceptron. On the left side we have the explanatory variables  $x_i$ , on the right side the outputs  $y_i$ , and in between we have several hidden units  $u_i$ , which are connected with each input and output. At each node the incoming values are weighted ( $w_{ij}$  and  $v_{ij}$ ) and summed up. At the hidden nodes we additionally apply a so-called activation function (usually  $1/(1 + e^{-\beta x})$ ), which lets the node function similar to a neuron in a brain: it will only fire an outgoing stimulus if the sum of incoming stimuli exceeds a certain threshold value.

The network is trained (i.e. the model is estimated) by means of back propagation, whereas numerous software libraries already exist that carry out these computations. The number of hidden units within a specific model is commonly determined by partitioning the available data set into three subsets. One is used for training, the second one is used for testing (and therefore for determining the

<sup>8</sup>I.e. a artificial neural network with a single hidden layer. Figure 2.2 is an example for such a single layer perceptron.

”optimal” number of hidden units), while the third one is used as a validation of the final model.

### Non-Parametric Estimation

Non-parametric estimation is generally based on a kernel estimation of the underlying density function<sup>9</sup>. Similar to Neural Networks, the model builder is not forced to determine, respectively assume structural relations a-priori. Accordingly this procedure also requires a lot of data, it will also only provide a good fit within the operating range of the available data, and furthermore suffers the *curse of dimensionality*<sup>10</sup>, which just allows us to model a very few number of explanatory variables.

Sales  $S$  are modelled as the conditional expected sales plus a random term ( $S = \mathbb{E}(S|x) + u$ ). In order to calculate the conditional expectation we first estimate the conditional distribution  $f_{S|x}$ , which is the ratio of the joint distribution  $f_{S,x}$  to the marginal distribution  $f_x$ . These distributions can be estimated by smoothing the histogram of the observations over the complete data space. This is done via a so called *kernel*, which basically calculates for every point in the space a weighted average of the number of observations within the ”near” distance<sup>11</sup>.

A quite common approach in marketing is also to use a semi-parametric approach, which could for example combine a parametric model for the structural relation with a non-parametric estimation for the random component.

### Structural Equation Models

Concluding this section we will now turn to Structural Equation Models (SEM), since its application has also been facing growing popularity for modelling sales functions in marketing science over the past two decades.

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<sup>9</sup>in our case of the sales function  $S$

<sup>10</sup>*Curse of dimensionality* denotes the phenomena, that with each extra variable an additional dimension is added to the data space, and therefore the amount of required observations grows exponentially.

<sup>11</sup>A crucial parameter within this process is the chosen *bandwidth* of the kernel, which determines the trade-off between the bias and the variance of our estimator.

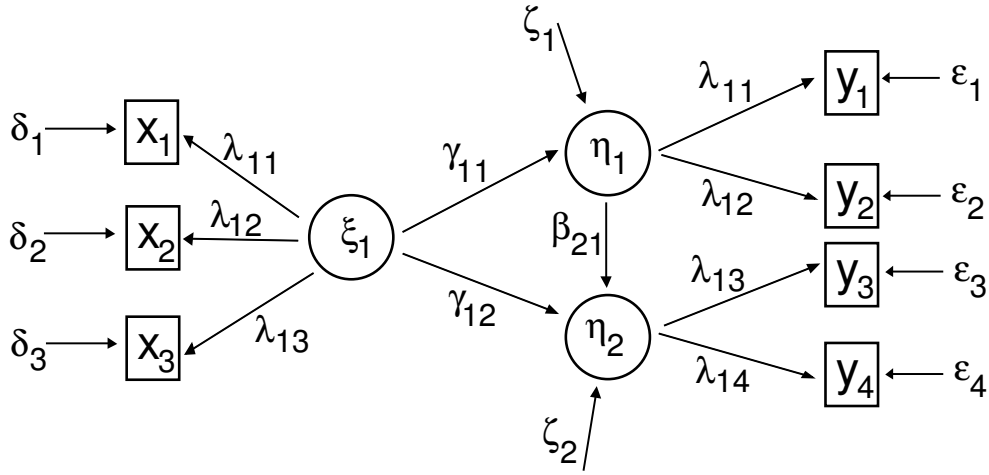


Figure 2.3: a structural equation model

SEM provide a framework which cleanly separates the underlying structural relations among latent (i.e. non-observable) variables from their actual measurement. Each of these latent variables is tried to be measured through a number of manifest (i.e. observable) variables, whereas this process is disturbed by exogenous errors.

In a first step the model builder determines (respectively guesses) the relevant latent and manifest variables and their causal ordering. This results in a corresponding path diagram, as it is illustrated in figure 2.3. The left side represents the inputs, respectively the right side the outputs. Each relation is represented by a path, whereas the direction of the path corresponds to the causal ordering.

Now, if we assume each relation to be of linear type, then we can derive the following equations:

$$\begin{aligned}
 \text{structural model:} \quad & \eta = B\eta + \Gamma\xi + \zeta \\
 \text{measurement model:} \quad & y = \Lambda_y\eta + \varepsilon \\
 & x = \Lambda_x\xi + \delta,
 \end{aligned}$$

$\eta$  and  $\xi$  are the latent variables, which are measured through the vectors  $x$  and  $y$ .  $\zeta$ ,  $\varepsilon$  and  $\delta$  denote the measurement errors. Overall we have eight param-

eter matrices (the four regression matrices  $B, \Gamma, \Lambda_x, \Lambda_y$  and the four covariance matrices  $\Phi, \Psi, \Theta_\varepsilon, \Theta_\delta$ ) which need to be estimated<sup>12</sup>. If all of these matrices were known then the joint covariance matrix  $\Sigma$  of  $x$  and  $y$  can be explicitly calculated. Therefore the estimation can be performed by choosing the eight parameter matrices such, that the resulting  $\Sigma$  is a best "fit" for the observed covariance matrix  $S$  from the sample. The actual distance measure will depend on the estimation method that is used (e.g. Maximum Likelihood or Generalized Least Square).

## 2.2 Sales Drivers

This section tries to identify several possible sales drivers, which are commonly operationalized and included into market response models in practice<sup>13</sup>.

### Advertising

Advertising subsumes a wide variety of different types of communication between companies and customers, whereas the variable of interest is generally the so called *advertising impact*, a latent variable which can not be measured directly. Therefore practitioners have to use other measures like *advertising expenditures* and *advertising exposure*<sup>14</sup>, or introduce intermediary variables (which need to be measured separately through consumer surveys or experiments) like *recall of advertising messages* and *brand awareness* to take the impact of advertising into account.

Furthermore it is up to the model builder whether to use aggregated advertising expenditures, or to include different advertising activities (e.g. TV commercials,

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<sup>12</sup>It should be clear that the stated model is due its large number of parameters highly non-identifiable, and therefore requires several additional constraints to become identifiable.

<sup>13</sup>the following classification is inspired by Hanssens, Parsons & Schultz [12] p.55ff

<sup>14</sup>Advertising exposure is generally understood as the number of times consumers are exposed to advertising, and is calculated as *reach* times *frequency* of an advertisement. Reach is the number of different people that are exposed to an advertisement within a certain period of time, whereas frequency is the average number of exposures within this period.

billboards,..) separately. Such a decision will depend on the availability of data and also on the primary focus of the research.

## **Pricing**

Price promotion is another marketing instrument commonly used by companies in order to push sales and therefore should also be considered and incorporated into the overall model.

Manufacturer promotions can take different kind of forms, including coupons, bonus offers, special refund offers, price packs and free samples.

## **Retail Distribution**

It should be clear that the sales of a consumer good are strongly related to the amount of stores at which they are available. Therefore retail distribution is taking into account, usually by calculating the percentage of retail outlets, which carry a certain brand, respectively a weighted percentage, which further represents the size (i.e. sales) of the stores adequately.

## **Retailer Merchandising**

Besides the marketing activities of the manufacturer, also retailers will try to promote their offered goods. Sales for a particular brand will therefore additionally depend on possible temporary price cuts, on the amount of occupied shelf space (and whether its placed at eye level, or above or below that) at the store and on the use of special displays at the point-of-purchase. Since these measurements take place on a per store, respectively on a per retail-chain basis, it is necessary to aggregate such activity to a common level for the whole market.

## **Personal Selling**

For consultancy-intensive products (e.g. industrial goods) and services (e.g. insurances), which require trained salespeople to sell them, the actual number and

the qualification of the salespeople, respectively the amount of customer contact time can be used as explanatory variables in the model.

## Product

The product itself is certainly also a significant sales driver, but the variety of properties (quality, color, weight, package,..) makes it difficult to operationalize it. Common measures would include the number of available package sizes, the number of variations of a product or the perceived quality by customers.

## Environmental Variables

But sales do not just depend on factors, that are under the direct control of a company, but also on a number of external factors. Just to list a few, this can include for example current interest rates, competitors marketing activities, tax burden or even weather conditions. It is the task of the practitioner to identify, measure and include the most relevant external factors into the model.<sup>15</sup>

Before proceeding we will shortly mention several further possibilities of *defining variables*. One-time or periodic events are usually modelled through the use of so-called *dummy variables*, which evaluate to 1 in case the event was present at a certain time, and 0 otherwise. Such dummy variables are used for example for incorporating additional purchases before Christmas or to represent a certain advertising campaign. In case that two factors are included via their ratio, we speak of *relative variables*. *Differenced variables* are generally used for taking dynamics into account, by building the difference between certain historic levels of a variable. *Stock variables* are another way of incorporating dynamics, which is done by aggregating the weighted past levels into one common variable.

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<sup>15</sup>Chapter 5 will be completely devoted to the adequate modelling competition.

## 2.3 Dynamics

The full impact of a change of a sales driver might not occur immediately (i.e. in the same observation period), but will still show significant impact later on (this phenomena is referred to as the *carryover effect*). One of the reasons for this is that customers, retailers and competitors actually need a certain time to react to a marketing activity (the so-called *delayed response effect*), and that these reactions might be more like a gradual adjustment, than an abrupt change. Sometimes people can even show a reaction in advance, i.e. *anticipate* an expected action. It is for these reasonings that market response models are generally required to incorporate dynamic effects appropriately, in order to provide an adequate representation of the market mechanism.

Especially the impact of advertising is considered to be a dynamic process. Brand awareness is for example the result of *all* past advertising efforts (and not just of the current ones), which will certainly decrease under the absence of it. Advertising is also a powerful instrument to establish *brand image* and to build *brand value*, and therefore is able to create long-term customer relationships. But on the other hand too much advertising can also diminish a customers receptiveness for new advertisement, and therefore future marketing expenditures might become less effective.

### 2.3.1 Lag Structure Models

A common practice is to incorporate advertising dynamics into a model by aggregating past advertising expenditures into one stock variable, which is then used in the overall model. On the one hand a stock variable (usually referred to as *adstock* for advertising) and its impact are easy to communicate to the management, on the other hand it simplifies the estimation since the dynamic effects are already subsumed into one variable.

Probably the most common type of lag structure is the *geometric distributed*

lag model:

$$Q_t = \beta_0 + \beta \sum_{k=0}^{\infty} \omega_k X_{t-k}$$

$$\omega_k = (1 - \lambda)\lambda^k, \text{ where } k = 0, 1, 2, \dots \text{ and } 0 < \lambda < 1$$

which can be estimated by applying the Koyck transformation which results in the estimation of a linear equation.

This lag structure has the drawback of not being able to represent any delayed response effect, since the biggest impact of an action is assumed to be immediate. Sometimes campaigns do not show any effects at all in the beginning, but will lead to purchases later on. In such cases a *negative binomial distribution* for the weights  $\omega_k$  could be used.

$$\omega_k = \frac{(r + k - 1)!}{(r - 1)!k!} (1 - \lambda)^r \lambda^k$$

$$\text{where } k = 0, 1, 2, \dots, 0 < \lambda < 1 \text{ and } r \in \mathbb{N}$$

But also a more general *polynomial lag structure* is feasible. In a study by Kamp & Kaiser on the New Yorker fluid milk market [17], which will be discussed later on in section 4.6, the authors use for example a quadratic lag structure to model the adstock.

### 2.3.2 Time-continuous vs. Time-discrete Models

The question on whether to use time-discrete or time-continuous models to represent dynamics will find contrary answers among researchers, mainly depending on their primary goal of research (e.g. whether it has an empirical or a theoretical focus):

*"In general, a continuous model and its discrete version may not lead to identical implications. This insight is important because empirical studies must be based on discrete models, whereas theoretical implications often are deduced from their continuous versions."* Park and Hahn [20] p.403

One of the primary drawbacks of time-discrete models is their dependency on the chosen (respectively given) length of the observation periods. For example empirical studies with annual data tend to show a significantly longer duration of advertising carryover effect than studies with monthly or weekly data do (more on this *data interval bias* can be found at Hanssens, Parsons & Schultz [12] p.174). But on the other hand the theoretical foundation of the estimation of time-discrete models is by far more advanced and developed than it is the case for time-continuous models.

In the following chapters the emphasis turns out to be more on time-continuous models. This seems reasonable considering that we are going to discuss several dynamic effects (competitive reactions, effects of pulsing policies,...) on a theoretical base, without having the need to support them by means of empirical estimations.

# Chapter 3

## Control Theory

After having identified market response as a thoroughly *dynamic* process in the last chapter, we will now turn towards control theory and its methods, in order to handle these dynamics appropriately.

Optimal control theory is about the optimal controlling of dynamic system with respect to a certain target measure. It will prove useful for gaining a *qualitative* insight in particular market mechanism, for understanding what is going on, rather than just delivering numerical solution to a particular decision problem.

We will present the maximum principle for the standard problem, which provides necessary conditions for a time-continuous dynamic system, then demonstrate its application on a simple advertising model, and follow with a general overview and categorization of existing advertising models.

### 3.1 Pontrjagin's Maximum Principle

The aim is to find for each  $t \in [0, T]$  optimal values for the control variables, which themselves underly certain restriction, so that the generated utility (resp. profit) over some time period becomes maximal.

The overall profit consists of the discounted accumulated utility flow and of

the discounted rest value at time  $T$ .

$$J = \int_0^T e^{-rt} F(\mathbf{x}(t), \mathbf{u}(t), t) dt + e^{-rT} S(\mathbf{x}(T), T) \rightarrow \max \quad (3.1.1)$$

$\mathbf{x}(t) \in \mathbb{R}^n$	... (vector) state variable
$\mathbf{u}(t) \in \mathbb{R}^m$	... (vector) control variable
$F$	... utility function, dependent on the current state, the control and time $t$
$S$	... rest value of the state $\mathbf{x}(T)$ at time $T$
$r$	... discount rate of the decision maker

The state of the system is known for time  $t = 0$  (*initial condition*), and its subsequent changes depend on the chosen control and are described via differential equations (*state equation*).

$$\mathbf{x}(0) = \mathbf{x}_0 \quad (3.1.2)$$

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t), \mathbf{u}(t), t) \quad (3.1.3)$$

The control itself generally underlies certain restrictions, which can depend on the current state and time, whereas Pontrjagin's theorem will make the simplifying assumption of a state- and time-independent constraint:

$$\mathbf{u}(t) \in \Omega \subseteq \mathbb{R}^m \quad (3.1.4)$$

In case that  $\mathbf{F}$  and  $\mathbf{f}$  are continuously differentiable with respect to  $\mathbf{x}$  and continuous w.r.t.  $\mathbf{u}$  and  $t$ , and further demand continuous differentiability from  $\mathbf{S}$  w.r.t.  $\mathbf{x}$  and  $T$ , then we refer to (3.1.1) - (3.1.4) as the standard control problem. An important special case of this is, where  $f$  and  $F$  do not depend on time  $t$  explicitly, which is denoted as an *autonomous* problem.

If we further define the (current-value) Hamilton function  $H$  as

$$H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) = F(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda(t)f(\mathbf{x}(t), \mathbf{u}(t), t), \quad (3.1.5)$$

whereas  $\lambda \in \mathbb{R}^n$  is the so-called *costate* (resp. the *adjoint state*) of the system, we can state the following necessary conditions for an optimal control path:<sup>1</sup>

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<sup>1</sup>taken and translated from Feichtinger [7]

**3.1.1 Theorem.** (*Maximum principle for the standard problem*)

Let  $\mathbf{u}^*(t)$  be an optimal control path for the standard control problem (3.1.1) - (3.1.4), and  $\mathbf{x}^*(t)$  the corresponding optimal state path, then there exists a continuous and piecewise continuously differentiable function  $\lambda(t) = (\lambda_1(t), \dots, \lambda_n(t)) \in \mathbb{R}^n$ , so that the following conditions are fulfilled:

The maximum condition: for each  $t \in [0, T]$ , where  $\mathbf{u}^*(t)$  is continuous

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda(t), t) = \max_{\mathbf{u} \in \Omega} H(\mathbf{x}^*(t), \mathbf{u}, \lambda(t), t), \quad (3.1.6)$$

the adjoint state equation:

$$\dot{\lambda}(t) = r\lambda(t) - H_x(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda(t), t), \quad \forall t \in [0, T], \quad (3.1.7)$$

and the transversality condition:

$$\lambda(T) = S_x(\mathbf{x}^*(T), T). \quad (3.1.8)$$

In order to solve a specific control problem it is therefore necessary to solve (3.1.6) for  $\mathbf{u}^*$  in dependency of  $\mathbf{x}^*$ ,  $\lambda$  and  $t$  analytically. Inserting this into (3.1.2), (3.1.3), (3.1.7), (3.1.8) will lead to a two-point boundary value problem, which is generally hard to solve explicitly.

Note, that the stated theorem only provides necessary conditions. In order to actually identify a control path as optimal, we need *sufficient* conditions, which can for example be found in Feichtinger [7] p.34, and are basically conditions regarding the concavity of the Hamiltonian.

Further it is quite common in economical context to rather optimize for an unlimited time. A maximum principle for unlimited time horizon can be found in Feichtinger [7] p.39f.

**3.1.1 Economic Interpretation**

The stated conditions of the maximum principle allow some immediate interpretation, which will already provide some insight into an optimal control in dynamic systems.

Let us define the maximized utility function as

$$V(\mathbf{x}, t) = \max_{\mathbf{u} \in \Omega} J(\mathbf{x}, t) = \max_{\mathbf{u} \in \Omega} \int_t^T e^{-rt} F(\mathbf{x}, \mathbf{u}, t) dt + e^{-rT} S(\mathbf{x}, T), \quad (3.1.9)$$

and refer to it as the *value function* of our system. Then it can be shown that the costate  $\lambda(t)$  equals the derivative of the value function with respect to the state.

$$\lambda(t) = \left. \frac{\partial V(\mathbf{x}, t)}{\partial \mathbf{x}} \right|_{\mathbf{x}=\mathbf{x}^*(t)} \quad (3.1.10)$$

For this reason  $\lambda$  is also known as the current-value *shadow price* of  $\mathbf{x}$ , since it represents the theoretical price the decision maker would be willed to pay for a marginal change in  $\mathbf{x}$ . The Hamiltonian (3.1.5) therefore can be seen as the sum of the direct and indirect impact of a chosen control  $\mathbf{u}$  at time  $t$ . The immediate benefit is represented by the generated utility  $F$ , whereas the indirect effect is the result of a change in the state (see state transformation (3.1.3)) weighted by its corresponding shadow price  $\lambda$ .

Therefore the maximum condition (3.1.6) basically states that the control variables have to maximize the Hamiltonian (i.e. the profit) at all times<sup>2</sup>.

## 3.2 A simple Advertising Model

$x(t)$	... number of purchased units
$u(t)$	... advertising rate
$c(u)$	... advertising costs
$\pi(x)$	... generated profit (excluding advertising costs)

We will assume a particular convex cost function ( $c(u) = \frac{1}{2}u^2$ ), and assume a strictly increasing, concave and continuously differentiable revenue function  $\pi$ . The number of purchases are assumed to decrease at a constant rate  $\delta$  and increase at the level of the advertising rate. Furthermore we try to find the optimal control

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<sup>2</sup>This result is of a similar character as a result derived in dynamic programming, which states that each partial sequence of an optimal sequential decision, has to be optimal on its own.

for an unlimited time horizon. Hence the problem can be stated as follows:

$$\max_{u \geq 0} \int_0^{\infty} e^{-rt} (\pi(x) - \frac{1}{2}u^2) dt \quad (3.2.1)$$

$$\dot{x} = -\delta x + u, \quad x(0) = x_0 \quad (3.2.2)$$

We obtain the Hamiltonian as

$$H(x, u, \lambda) = \pi(x) - \frac{1}{2}u^2 + \lambda(-\delta x + u) \quad (3.2.3)$$

After confirming that the state transformation ( $f(x, u) = -\delta x + u$ ) and the profit rate ( $F(x, u) = \pi(x) - \frac{1}{2}u^2$ ) are both continuously differentiable, we can obtain possible candidates for an optimal control path by using Pontrjagin's maximum principle.

The maximum condition yields

$$(H_u = 0 \text{ for } u > 0) \quad \vee \quad (H_u \leq 0 \text{ for } u = 0)$$

Since in our model the marginal costs for advertising are 0 for  $u$  being 0 (i.e.  $c'(0) = 0$ ), we can exclude the second case, since any increase in  $u$  would already yield a higher profit. Hence

$$H_u = -u + \lambda = 0 \quad \Rightarrow \quad u = \lambda.$$

The costate equation (3.1.7) yields

$$\dot{\lambda} = r\lambda - H_x = r\lambda - \pi'(x) + \delta\lambda$$

Inserting  $u = \lambda$  into the state and costate differential equations yields the following canonical system:

$$\begin{aligned} \dot{x} &= & -\delta x & + \lambda \\ \dot{\lambda} &= & \underbrace{-\pi'(x)}_{\text{Nonlinearity}} & + (r + \delta)\lambda \end{aligned}$$

By determining the isoclines and the unique equilibrium and by evaluating the eigenvalues of the Jacobi matrix  $A$  at that point, we can sketch the state-costate

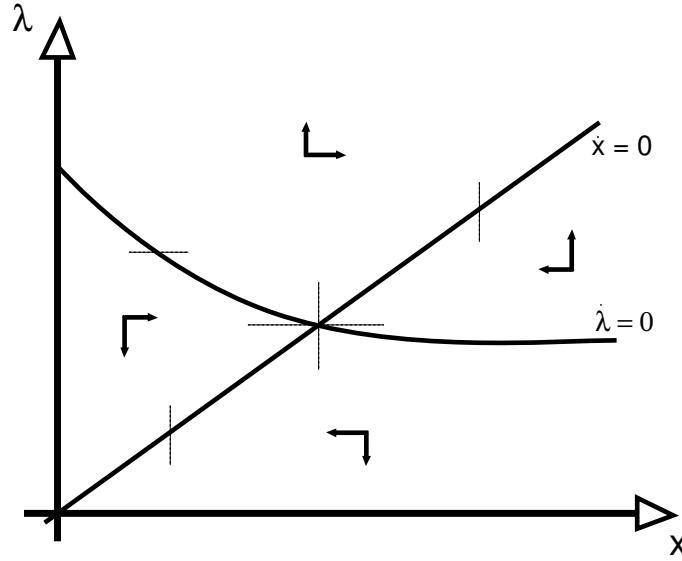


Figure 3.1: state-costate phase diagram

phase diagram (figure 3.2), which will provide qualitative insight into the solution path. The equilibrium:

$$\left. \begin{aligned} \dot{x} = 0 &\Rightarrow \lambda = \delta x \\ \dot{\lambda} = 0 &\Rightarrow \lambda = \frac{1}{r+\delta}\pi'(x) \end{aligned} \right\} \Rightarrow (\hat{x}, \hat{\lambda})$$

The Jacobi matrix A:

$$A = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial \lambda} \\ \frac{\partial \dot{\lambda}}{\partial x} & \frac{\partial \dot{\lambda}}{\partial \lambda} \end{pmatrix}_{(\hat{x}, \hat{\lambda})} = \begin{pmatrix} -\delta & 1 \\ -\pi''(\hat{x}) & r + \delta \end{pmatrix}$$

The eigenvalues  $\mu_1$  and  $\mu_2$ :

$$\begin{aligned} \det(A - \mu I) &= (-\delta - \mu)(r + \delta - \mu) + \pi''(\hat{x}) \\ &= \mu^2 + r\mu - \delta r - \delta^2 + \pi''(\hat{x}) = 0 \\ \Rightarrow \mu_{1,2} &= \frac{r}{2} \pm \underbrace{\sqrt{\frac{r^2}{4} \underbrace{-\pi''(\hat{x})}_{>0} + \underbrace{\delta r + \delta^2}_{>0}}}_{> \frac{r}{2}} \end{aligned}$$

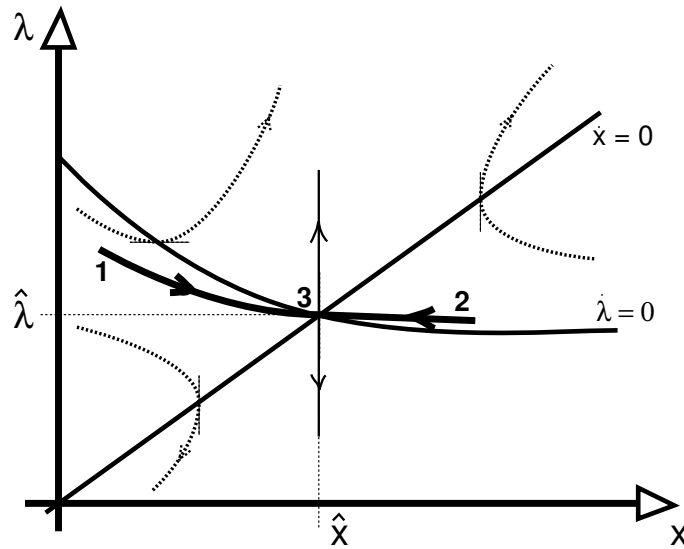


Figure 3.2: state-costate phase diagram

Since the expression under the square root will always be nonnegative and greater than  $\frac{r}{2}$ , two real eigenvalues are derived, one positive and one negative. Therefore the equilibrium  $(\hat{x}, \hat{\lambda})$  represents a *saddle-point* with a converging and a diverging direction.

Analyzing  $\dot{x}_x$ , respectively  $\dot{\lambda}_\lambda$  reveals the orientation of these paths, and will allow us to draw figure 3.1.  $\dot{x}_x = -\delta < 0$  implies that  $\dot{x}$  is decreasing from left to right in our diagram. Therefore  $\dot{x}$  has to be greater than 0 on the left side of  $\dot{x} = 0$ , and less than 0 on the right side. A similar argumentation can be used for  $\dot{\lambda}_\lambda = r + \delta > 0$ .

The transversality condition for an unlimited time horizon rules out all control paths which result in unbounded and negative-valued state-, respectively costate-paths<sup>3</sup>. Therefore we derive only three possible optimal paths, which are highlighted in diagram 3.2.

Together with the phase diagram we are able to deduct qualitative results from our model:

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<sup>3</sup>compare Feichtinger [7] p.43

**Path 1:**  $x(0) < \hat{x}$ , i.e. the initial value of purchases is relatively small, then  $\lambda(0)$  is rather large, and accordingly also the optimal advertising rate. As more and more customers are gained,  $\lambda(t)$  (the shadow value of a single extra purchase) decreases, since we have a concave profit function  $\pi(x)$ . Accordingly the optimal advertising policy will start high, but decreases towards its long-term equilibrium.

**Path 2:**  $x(0) > \hat{x} \Rightarrow \lambda(t)$  and accordingly  $u(t)$  are small in the beginning, but are increased towards the equilibrium. Therefore in this case we have a strictly increasing advertising policy as an optimum.

**Path 3:**  $x(0) = \hat{x} \Rightarrow \lambda(0) = \lambda(t) = u(t) = \hat{\lambda} \forall t$ , i.e. the optimal advertising is constant and the state remains in its equilibrium.

Finally we will perform a *comparative static analysis* regarding our parameters  $\delta$ ,  $r$  and  $\pi'$ :

**the sales decay rate  $\delta$ :**  $\frac{\partial \hat{x}}{\partial \delta} < 0 \wedge \frac{\partial \hat{\lambda}}{\partial \delta} > 0 \Rightarrow$  if the decay rate is rather high, i.e. the customers are not loyal and switch brands regularly, than the optimal advertising rate must be higher, whereas the achieved optimal customer stock will be lower in the equilibrium.

**the discount rate  $r$ :**  $\frac{\partial \hat{x}}{\partial r} < 0 \wedge \frac{\partial \hat{\lambda}}{\partial r} < 0 \Rightarrow$  a higher discount rate obviously decreases the value of a large customer base, which is reflected in these derivatives appropriately. A higher discount rate implies a smaller optimal  $\hat{x}$  and also a smaller optimal advertising policy  $\hat{u} = \hat{\lambda}$ .

**the gradient of the profit function  $\pi(x)$ :**  $\frac{\partial \hat{x}}{\partial \pi'} > 0 \wedge \frac{\partial \hat{\lambda}}{\partial \pi'} > 0 \Rightarrow$  the higher the profit per purchase will be, the higher will also be the optimal number of purchases and the optimal advertising rate.

### 3.3 General Overview of Dynamic Optimal Control Models in Advertising

The following section will present a classification of current control models which has been carried out by Feichtinger, Hartl & Sethi in [8].

#### 3.3.1 Capital Stocks Generated by Advertising, Price and Quality

The main idea behind the models within this category is, that marketing activities are just like any other investment, and as such are able to accumulate capital stocks. This approach originates in a well-known article by *Nerlove & Arrow*, published in 1962 [19], who have been the first to introduce a variable representing advertising stock, the so-called *goodwill*. Generated sales ( $s$ ) are assumed as a function of this goodwill ( $A$ ), rather than of the current advertising ( $u$ ). Like any other stock, it will depreciate at some certain rate ( $\delta$ ) over time, if nothing is invested anymore. This yields the following central relation<sup>4</sup>:

$$\begin{aligned}\dot{A} &= u - \delta A \\ A(0) &= A_0\end{aligned}$$

Such a model results in a so-called *bang-bang policy*, meaning that there exists an optimal level  $\hat{A}$  which is tried to be reached as quickly as possible. If the current goodwill is below this level, it is optimal to advertise at maximum level, whereas if it is above, then no advertising should be carried out until goodwill reaches its optimal level<sup>5</sup>.

Several extensions have evolved since the original article, which basically try to model the goodwill accumulation more sophisticated. One approach even tries to incorporate two different distributions of time-lags, one for reacting to an advertising and for forgetting an advertising. But there are also critical comments regarding this trend: Bultez and Naert note that "*the current tendency*

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<sup>4</sup>expressed in continuous time as a differential equation

<sup>5</sup>see [8] p.199 for references to literature regarding this result

*to build and estimate increasingly sophisticated lag models does not seem totally justified.*"<sup>6</sup>

The more promising extensions to the Nerlove & Arrow model seem to be, when also other marketing activities like pricing, or especially the quality of a product are allowed to generate stocks (e.g. *reputation*).

### 3.3.2 Sales-Advertising Response Models

These models have in common, that sales are directly put into relation to advertising via a differential equation. Again, this approach originates in a specific paper, this one being published by *Vidale & Wolfe* in 1957 [25].

Vidale & Wolfe explicitly take into account that advertising shows decreasing marginal returns for an increasing customer base. They do so by stating the following relation:

$$\dot{x} = \rho u(1 - x) - \delta x, \quad x(0) = x_0,$$

whereas  $x$  denotes the current fraction of the overall market potential,  $\rho$  the advertising effectiveness and  $\delta$  the sales decay rate (whereas all of them are assumed to be constant). Higher sales figures therefore imply, that the remaining target group for advertising (i.e. the remaining market share  $1 - x$ ) becomes smaller and therefore advertising becomes less effective.

Feichtinger, Hartl and Sethi come in their article to the general conclusion, that *"sales-advertising response models [...] can be considered more realistic than the capital stock models [...] since it is certainly more easy to measure, estimate, or even define sales compared to 'goodwill'"*.

### 3.3.3 Cumulative Sales or Market Growth Models

This category of models emphasizes on the significance of cumulated sales, and are therefore more suitable to model changes during a product life cycle appropriately.

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<sup>6</sup>see [8] p.200 for this quote

On the one hand the cumulated sales can have a positive carry-over effect, if there is a positive word-of-mouth recommendation among customers, and there might be negative carry-over effects which are due to a saturated market. Furthermore a so-called *cost learning phenomenon* usually takes place with an increasing number of produced goods, which states that companies face decreasing production costs per unit over time (due to optimization or technological progress for example).

### 3.3.4 Models with more than one State Variable in the Advertising Process

Whereas the heading of this category might irritate, it becomes clear when considering that pulsing policies are *only* able to turn out optimal if at least two state variables are included into the control model. Therefore this category could be also named pulsing models, to which we will devote the complete next chapter in this thesis.

### 3.3.5 Interaction with other Function Areas

Due to their increasing complexity this category is still rather young and evolving. These model try to a certain extent to link decisions regarding marketing, production, finance and personnel into one overall model, which takes their interdependencies into account.

Just to give an example, a new advertising campaign might be necessary according to the marketing department, but might finally prove to be counterproductive in case that the production department is not able to satisfy the increased demand on time.

But be aware, that due to complex nature of such models *it is usually not possible anymore to derive general qualitative results such as monotonicity*<sup>7</sup>. All we can expect therefore are numerical results for specific models, estimated via real data.

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<sup>7</sup>see [8] p.216

### 3.3.6 Competitive Models

All of the models so far have (silently) assumed a monopolistic market, which is generally not the case, but is rather assumed for reasons of simplicity. There are several ways to incorporate competition, starting from including aggregated market activities into the model (e.g. overall advertising expenditures in a market), over modelling competitors as passive reactors (via the use of *reaction functions*), to more advanced game-theoretic approaches, which try to model all competitors as *individual optimizers* with differing preference functions on their own. Due to the high relevance (but low spread) of such models we will devote the complete chapter 5 on competition.

# Chapter 4

## Pulsing

”There is empirical evidence in marketing that pulsing advertising policies may be more effective than equal spending of advertising budget.” [6] p.326

”Empirical evidence indicates that a given moderate number of ads per year may achieve higher average effect when concentrated in flights than when spread equally.” [5]

”Advertising practitioners often believe that pulsing can be superior to the even strategy. The evidence supporting this belief has been reported from field experiments [..], laboratory experiments [..], and computer simulations [..].” [20]

Considering these quotations it seems hardly understandable that the number of models which are able to produce a pulsing policy as optimum are still sparse and that the question of which factors actually lead to pulsation still remains unanswered for the most part.

We will devote this whole chapter solely to pulsation, and try to give a historical overview of the progress regarding this issue in marketing science.

Before starting out, we should note that we explicitly do not intend to analyze periodic behavior which results from exogenously determined fluctuations (e.g.

seasonal demand or periodic boundary conditions), but try to find structural inherent characteristics of a model which will generate these cycles<sup>1</sup>.

## 4.1 Rao 1970

One of the first attempts to show the superiority of pulsing vs. even spending has been made by Rao in chapter 5 of "Quantitative Theories in Advertising" [21]. Rao develops a highly complex time-continuous model which incorporates different levels of loyalty among consumer groups, and models distinctly *switching* and *a change in primary demand* in respect to advertising expenditures for each of these groups<sup>2</sup>. Rao establishes via his model, that the superiority of pulsation depends on the one hand on the relative weights of the switching effects and of the change in consumption effects, and on the other hand on the companies market share. If switching effects dominate for example, then it is, according to Rao, just profitable for the company with a small market share to pulse, since the number of customers who will eventually switch toward the company in times of high advertising, should exceed the number of people who switch away in the times of low advertising.

At first sight it is not clear which one of the numerous assumption, on which this model is based on, actually lead to pulsation, but an article by Sasieni on "Optimal Advertising Expenditure" published in 1971 [22] seems to clarify this.

## 4.2 Sasieni 1971 [22]

Sasieni derives that in case of a nonconcave sales response function (e.g. S-shaped) a chattering policy becomes optimal when operating in the convex part. A chattering control is a policy which switches between two levels in infinitesimally short time back and forth. By looking at figure 4.1 this result seems to be

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<sup>1</sup>see Feichtinger [6] p.313

<sup>2</sup>An exhaustive discussion (respectively critique) on this particular model can be found in Appendix 4.7.

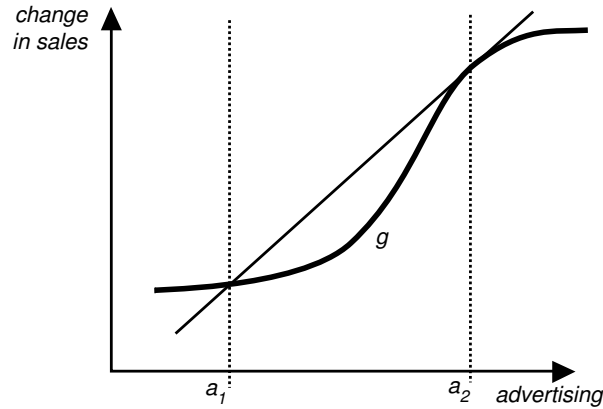


Figure 4.1: nonconcave response function

reasonable. If we assume a fixed advertising budget  $a_0$  which lies on the convex part of the response function  $\dot{s} = g(s, a, t)$  (i.e.  $\frac{\partial g}{\partial a} \geq 0$ ,  $a_1 \leq a_0 \leq a_2$ ), then it turns out to be more profitable to operate on the straight line connecting  $g(a_1)$  and  $g(a_2)$  by using a *mixed policy*  $ua_1 + (1 - u)a_2$  with  $0 \leq u \leq 1$  than spending a constant amount of  $a_0$ . In order to approach this mixed policy at all time, we have to iterate in infinitesimally short intervals.

*"In practice, the mixed policy cannot be followed because discrete changes in expenditure levels cannot be made too frequently. When a mixed policy is optimal the best we can achieve is to use a cyclic policy in which we advertise for short intervals at each of the appropriate levels." [22]*

This means that it is practicably impossible for an advertiser to alternate his expenditures between high and low levels in an arbitrarily short period of time, and a pulsing policy is assumed to be the best approximation for chattering. This reasoning has been commonly used in the past as a justification for the occurrence of pulsation policies.

It should be clear, that such a model is not consistent with reality, since a single customer is practically *not* able to distinguish between an even and a chattering

policy, and therefore the outcome for the company should be the same. In order to overcome this drawback we have to model a minimum length of time until a pulse is actually recognized as a pulse, or more sophisticated, we need to model an adaption process for advertising perception.

On the other hand, Sasieni results further suggest that *in the case of a concave or a linear response functions, pulsing will always be inferior compared to even spending!* But since Sasieni assumed a symmetric response function and a model with a single state variable this result will not collide with later findings in this chapter.

### 4.3 Simon 1982: ADPULS [24]

In a paper published in 1982 Hermann Simon was able to show the superiority of a pulsing policy by using an *asymmetric sales response function* in a dynamic time-discrete model.

The asymmetry is a result of incorporating *advertising wearout* into the model, i.e. the commonly observed phenomena that an increase in advertising leads to an immediate sharp increase in sales, which then subsequently fall off over time, even if the higher advertising level is maintained (see figure 4.2). On the other hand, with a persistent reduction in advertising, we can usually observe a gradually decrease in sales level until it reaches its new equilibrium level.

A large amount of existing models is not able to represent this asymmetric behavior, which is not just asymmetric in magnitude, but also regarding the functional shape, correctly, and therefore these are, according to Simon, *structurally misspecified* on an *a priori* basis.

A number of explanations exist to justify the wearout phenomena. One argument is that customers tend to try out products which are promoted, but that a possible dissatisfying user experience might prevent further purchases. The argumentation used by Simon for his particular model, is based on the general *adaption level theory*. Its application on marketing implies that we are dealing with two different stimuli, a *stimulus level* and a *stimulus differential*, which both

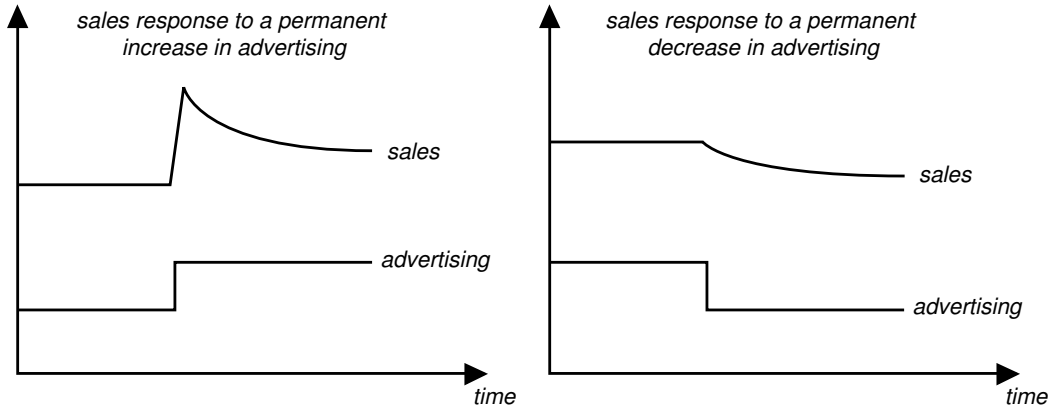


Figure 4.2: advertising wearout [24]

have to be incorporated separately. The stimulus level solely depends on the current advertising level, whereas the stimulus differential is a function of current advertising in relation to past advertising.

Since "the appearance of an advertisement is more likely to be perceived than its absence" Simon neglects the stimulus differential in the case of a decreasing advertising level, and therefore establishes the following general time-discrete asymmetric response model:

$$q_t = \underbrace{f(A_t, \bar{A}_t, q_{t-1})}_{\text{stimulus level}} + \underbrace{\max\{0, g(\Delta A_t)\}}_{\text{stimulus differential}},$$

- $q_t$  ... sales volume, or market share in period  $t$
- $A_t$  ... advertising expenditures in period  $t$
- $\bar{A}_t$  ... (aggregated) advertising expenditures of the competitors
- $\Delta A_t$  ... either the absolute or relative difference between  $A_t$  and  $A_{t-1}$   
 i.e.  $A_t - A_{t-1}$  or  $(A_t - A_{t-1})/A_{t-1}$

The crucial assumption which is made here is, that the stimulus differential is defined in relation to the advertising efforts of the *previous* period.

In his paper Simon used the following particular model to estimate models and to obtain optimal policies analytically:

$$q_t = a + \lambda q_{t-1} + b \ln A_t + c \max\{0, \Delta A_t\}, \tag{4.3.1}$$

whereas  $a$ ,  $\lambda$ ,  $b$  and  $c$  are the parameters to be estimated.  $f$  has been chosen to be the logarithmic function in order to incorporate diminishing marginal returns on advertising. Simon assumed constant prices, constant marginal cost, and an unlimited time horizon, which yields the following objective function:

$$\Pi_t = \sum_{\tau=0}^{\infty} [(p - C')q_{t+\tau} - A_{t+\tau}]z^\tau, \quad (4.3.2)$$

where  $z$  stands for the discount factor. In order to determine the optimal future advertising policy ( $A_{t+\tau}$ ,  $\tau \geq 0$ ) Simon proceeded by setting the derivative  $\frac{\partial \Pi_t}{\partial A_t} = 0$  and derives a pulsing policy as optimal<sup>3</sup>.

Using the  $Z$ -Transformation<sup>4</sup> equation 4.3.1 can be transformed to

$$(1 - \lambda Z)q_t = a + b \ln A_t + c \max\{0, \Delta A_t\}$$

$$q_t = \frac{a}{1 - \lambda} + b \sum_{j=0}^{\infty} \lambda^j \ln A_{t-j} + c \sum_{j=0}^{\infty} \lambda^j \max\{0, \Delta A_{t-j}\},$$

which reveals that the sales  $q_t$  are basically the sum of the exponentially smoothed past advertising levels  $A_t$  and the exponentially smoothed past (positive) advertising differences<sup>5</sup>. Intuitively it is therefore clear that an optimal policy tries to generate as many pulses as possible, whereas the magnitude of the pulses are either limited by a constrained advertising budget, or by the diminishing returns on advertising.

The optimal policy derived by Simon implies that advertisers should switch from low advertising to high advertising in each time period. This result has the obvious major drawback for the practitioner, that it is highly dependent on the length of the chosen time period. Basically Simon's result seems to be a discrete version of an optimal chattering control, which can not, as has been argued before, be a reasonable strategy. The crucial assumption, which leads to this result is, that advertisers are able to achieve a stimulus differential in every period, since

<sup>3</sup>see Appendix 4.8 for a further mathematical discussion of Simon's optimization

<sup>4</sup> $Z$  denotes the Shift-Operator, which maps  $(q_t)$  to  $(q_{t-1})$

<sup>5</sup>Note, that it is not guaranteed that wearout takes place like it is sketched in figure 4.2, since the peak does not necessarily exceed the new long-term equilibrium, if  $c$  is small enough.

the last advertising level has been taken as an anchor value. A more advanced model will therefore require to model the anchor value (or also called the *adaption level*) in a more sophisticated way. Luhmer et al. present such a model six years later.

#### 4.4 Luhmer et al.: ADPULS in continuous time [5]

In order to avoid a dependency on the chosen period length Luhmer et al. reformulated the ADPULS model for continuous time.

One of the drawbacks we recognized, while looking at models that yielded chattering controls, was the questionable assumption that the perception of advertising is solely a function of the *current* advertising spending. We rather want to model advertising effects as a sum of past advertising efforts, whereas more recent advertising would have a bigger impact than advertising further lying in the past. A simple way to do so, is to define  $A(t)$  (the currently effective advertising) as an exponential smoothing of past advertising levels  $u(t)$ .

Since in a time-continuous model we can not define a "last" advertising level anymore, we also have to modify the original ADPULS model with regard to the stimulus differential. The authors therefore define an adaption level  $S(t)$ , which functions as anchor value, by exponentially smoothing over the past effective advertising levels  $A(t)$ .

The time-continuous version of Simon's dynamic sales response function is of the following form

$$\dot{Q}(t) = f(A(t)) - \delta Q(t) + \max\{0, \omega(A(t) - S(t))\},$$

with  $Q(t)$  denoting the sales level at time  $t$  and  $\dot{Q}(t)$  the marginal change in  $Q(t)$  at time  $t$ . So, we again assume asymmetric behavior by modelling a differential stimulus, which just occurs in the case that effective advertising is above its long run average. For reasons of simplicity competitive advertising is discarded.

We further assume constraint advertising efforts  $u(t)$ , assume again a fixed gross profit  $\pi$  per sold unit, linear advertising costs and will choose specific functions  $f$  and  $\omega$  similar to the original ADPULS-model. The resulting model is therefore of the form:

$$\begin{aligned} \dot{Q} &= \underbrace{b \ln(A+1)}_{f(A)} - \delta Q + \max\{0, \underbrace{(A-S)\bar{w}}_{=\omega(A-S)}\} \\ \dot{A} &= u - \alpha A \\ \dot{S} &= (A-S)\gamma \\ &Q(0), A(0), S(0) \text{ fix} \end{aligned}$$

We are dealing with an optimal control problem with three state variables,  $Q$ ,  $A$ ,  $S$ , and one control,  $u$ . Due to the modelled asymmetry we have a kinked state equation, and therefore have to use the general maximum principle to derive necessary conditions for a solution.

The maximization of the current value Hamilton function

$$H = \pi Q - cu + \lambda(b \ln(A+1) - \delta Q + \max\{0, (A-S)\bar{w}\}) + \mu(u - \alpha A) + \nu(A-S)\gamma$$

results in an optimal control

$$u = \begin{cases} 0 & \text{for } \mu < c \\ \text{undefined} & \text{for } \mu = c \\ \bar{u} & \text{for } \mu > c. \end{cases}$$

*This means that if pulsation turns out to be optimal, then the optimal advertising level switches between zero and the upper boundary!* Note, that we did not restrict the advertising policy to this particular form a priori. Any kind of a *weaker* pulsation, with a non-zero lower advertising level, or with a continuous switch from high to low and low to high, would turn out suboptimal in this model, which is a highly interesting result for advertising practitioners. Looking at the derivative of the Hamilton function with respect to  $u$  reveals that this result

is robust against all kind of sales response functions (assuming that pulsing is superior to an even policy), but might not hold for other than linear advertising cost functions.

Together with the three adjoint state equations a boundary value problem with six differential equations is derived (see the referred paper for further details), which can be solved numerically for given (resp. estimated) parameter settings  $b, \pi, c, \alpha, \delta, \gamma, \bar{w}, r$  and  $\bar{u}$ . Luhmer et al. assume specific values for these nine parameters for which *a periodic solution turned out to be optimal*. Its trajectories and time-paths are sketched in the following figures. Figure 4.3 displays several different optimal trajectories for different start levels  $A(0)$  and  $S(0)$ , whereas all optimal solutions converge very fast (i.e. within the first cycle) towards the closed orbit, which is the optimal long-term policy. Figure 4.4 shows the optimal long-term paths of advertising efforts  $u(t)$ , effective advertising  $A(t)$  and of the adaption level  $S(t)$ . In order to understand the periodic cycles we divide the limit cycle into four sections:

**Phase 1:**  $A > S, A \nearrow, S \nearrow, \mu > c, u = \bar{u}, \omega = (A - S)\bar{w}$

$S$  is relatively small, so we will be able to achieve a differential stimulus. Furthermore, since the actual cost  $c$  of advertising is below the shadow price  $\mu$  of  $A$  (i.e. what we are willing to pay for an extra unit of effective advertising), we will advertise at the maximum level. With  $u$  being at maximum level, we have increasing  $A$ , but also increasing  $S$ , and therefore we will reach a point when the benefits of the differential stimulus are not high enough anymore. This occurs as soon as the shadow price drops below the actual costs  $c$ .

**Phase 2:**  $A > S, A \searrow, S \nearrow, \mu < c, u = 0, \omega = (A - S)\bar{w}$

Advertising is stopped, but since  $S$  is still larger than  $A$  for a short period of time, we still have a differential stimulus effect.

**Phase 3:**  $A < S, A \searrow, S \searrow, \mu < c, u = 0, \omega = 0$

After having stopped with advertising, the adaption level  $S$  recovers to a lower level again and we will reach a point, where it pays off again to advertise. This occurs as soon as the shadow price surpasses the actual costs  $c$ .

**Phase 4:**  $A < S, A \nearrow, S \searrow, \mu > c, u = \bar{u}, \omega = 0$

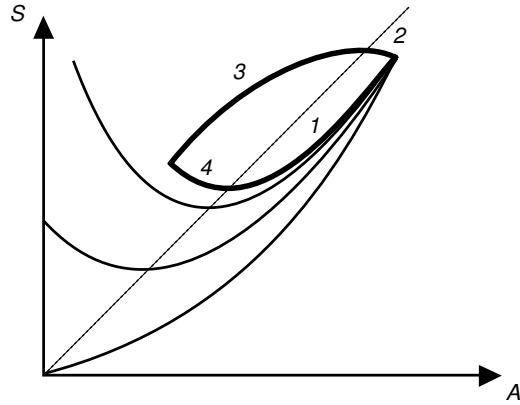


Figure 4.3: The ADPULS cycle in the  $(A,S)$ -state phase diagram.

In Phase 4 we start advertising again, but due to its delayed effect on  $S$ , the adaption level will remain its decrease for a littler longer. Note that we advertise although no differential stimulus currently occurs.

The major achievement of the time-continuous reformulation of Simon's ADPULS-model is the establishment of a periodic solution while we do not restrict the shape of the control a priori (like it has been done in Rao [21] or Simon [24]).

It seems as if the asymmetric shape of the sales response function is responsible for the occurrence of a cyclical behavior, but to this point no extensive sensitivity analysis regarding the parameters, resp. the underlying assumptions of the model has been carried out and published (neither numerically, nor analytically via the use of the Hopf bifurcation theorem).

## 4.5 Conclusion

Two key characteristics of the sales response function have been identified so far which might lead to pulsation. One is non-concavity and the second is asymmetry, whereas non-concavity has hardly any empirical justification. Hanssens and Parsons note: *"The preponderance of empirical evidence favors the strictly*

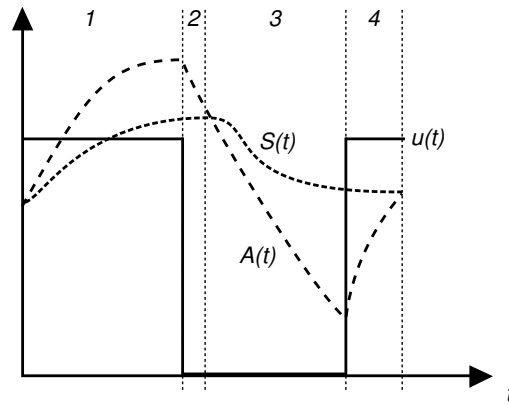


Figure 4.4: The optimal time path of  $u$ ,  $A$  and  $S$ .

*concave sales response to nonprice marketing decision variables.*" [4] p.437. Is Asymmetry the only remaining cause for pulsation? If we reject a test on asymmetry, does that automatically imply that even spending is superior? The answer seems to be NO.

G. Feichtinger and A. Novak [9] for example used a completely different approach which resulted in a pulsing strategy in advertising by incorporating diffusion, i.e. the interactions of buyers with potential buyers (=word-of-mouth recommendations). The flows between these two groups has been modelled via differential equations, whereas the company could influence these flows through advertising spending. For certain parameter constellations a cyclical solution could be established via the Hopf bifurcation theorem.

Yet in another paper by Hahn and Hyun [11] the authors show that the interaction of fixed and pulsing costs can make pulsing optimal (under reasonable conditions).

The *Hopf bifurcation* theorem provides a method of how to determine *critical parameter values which lead to limit cycles*. In an excellent paper by G. Feichtinger on "Limit Cycles in Dynamic Economic Systems" [6] various application of the Hopf bifurcation are presented and carried out (either analytically or numerically). Feichtinger concludes on page 341 that the following mechanisms

might generate a cyclical optimal behavior<sup>6</sup>:

- Non-concavities in the profit (utility) function or/and in the system dynamics.
- A "high" discount rate.
- Intertemporal substitution effects, e.g. adjacent complementarity in habit formation

Especially after studying "Persistent Oscillations in a Threshold Adjustment Model" by G. Feichtinger and A. Novak [10], which focuses on behavioral models of habit formation, the last item seems to provide a solid explanation for the occurrence of pulsation in the continuous ADPULS model<sup>7</sup>. It is the positive effect of advertising efforts on effective advertising and the negative effect on the adaption level that act here complementary in direction and different in time, and which might be responsible for the occurring pulsation. Due to its high complexity the continuous ADPULS model (with six differential equations!) has not been subject of an analytical sensitivity analysis via Hopf bifurcation, but this approach would/should definitely deserve some further investigation.

## 4.6 Framework by Kamp & Kaiser

In this final section of this chapter on pulsing we are going to present a framework published by Philip R. Vande Kamp and Harry M. Kaiser which practitioners might find useful for determining an optimal temporal advertising strategy while incorporating an asymmetric sales response function.

For an application of this framework on the generic fluid milk market in New York City see the referred papers [16] and [17].

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<sup>6</sup>One of the *necessary* conditions for cyclical optimal behavior is that we include at least two state variables in our model, since Hartl proved in 1986 that "the optimal state trajectory in one-dimensional autonomous control problems is always monotone" [24] p.171.

<sup>7</sup>rather than the asymmetric kinked sales response function

We assume a constant available advertising budget for each period (which might be set to the average historical level), assume constant gross profit per sold unit and will optimize over an infinite time horizon.

$$\begin{aligned} & \max_{a_t, \tau \geq t} \sum_{\tau=t}^{\infty} z^{\tau} p q_{\tau} \\ & q_t = f(a_t, \mathbf{a}_{t-1}, \mathbf{W}_t) \\ & s_{t+1} = (1+r)s_t + b - a_t, \quad s_t \geq 0 \\ & 0 \leq a_t \leq \bar{a} \end{aligned}$$

$z$	... discount rate
$p$	... gross profit per sold unit
$q_t$	... number of sold units in period $t$
$a_t$	... advertising expenditures in period $t$
$\mathbf{a}_{t-1}$	... vector of past advertising expenditures, i.e. $(a_{t-1}, a_{t-2}, a_{t-3}, \dots, a_{t-n})$
$s_t$	... available funds for advertising
$b$	... fixed level of funds provided for advertising in each period
$r$	... interest rate for savings
$\bar{a}$	... upper constraint for advertising budget per period
$\mathbf{W}_t$	... vector of factors other than advertising

We assume that the portion of  $b$  which is not spend on advertising during a period is put aside, and is available (together with interest) in future periods.

In a first step the sales response function needs to be estimated via econometric methods, and in a second step the optimal expenditures are determined by *successive approximation*, a common technique of Operations Research.

In their particular model the authors used the following specific form for the sales response function, whereas practitioners can certainly build their own while the remaining optimization techniques will still remain valid: Past advertising levels are incorporated via an Ad-Stock variable  $A_t = \sum_{s=0}^n w_s a_{t-s}$ . In order to reduce the number of parameters, the weights have been assumed to follow a quadratic exponential form  $w_s = e^{\varphi_0 + \varphi_1 s + \varphi_2 s^2}$ . Asymmetry has been modelled by

introducing additional variables for the sales response function<sup>8</sup>:

$$Z_{t-i}^I = \max\{\ln(A_{t-i}/A_{t-i-1}), 0\}, \quad i = 0, \dots, m$$

$$Z_{t-i}^D = \min\{\ln(A_{t-i}/A_{t-i-1}), 0\}, \quad i = 0, \dots, m$$

The final function is a mix between a multiplicative model and an exponential model and has the form:

$$\ln(q_t) = \beta \ln(A_t) + \sum_{i=0}^m \alpha_i^I Z_{t-i}^I + \sum_{i=0}^m \alpha_i^D Z_{t-i}^D + \Phi \mathbf{W}_t$$

After estimating the parameters by a combination of *Ordinary-Least-Squares* and *grid search*<sup>9</sup>, we can turn towards optimization.

If we denote the maximized object function with

$$\nu(s_t, \mathbf{a}_{t-1}) = \max_{a_t, \tau \geq t} \sum_{\tau=t}^{\infty} z^\tau pq_\tau,$$

then, applying Bellmann's function equation, we can reformulate the so called *value function* to

$$\nu(s_t, \mathbf{a}_{t-1}) = \max_{a_t} \{pq(a_t, \mathbf{a}_{t-1}, \mathbf{W}) + z\nu(s_{t+1}, \mathbf{a}_t)\}$$

As soon as we can determine the value function  $\nu$ , we also can derive the corresponding policy function  $h$ , which maps each start value  $(s_t, \mathbf{a}_{t-1})$  to an optimal advertising level  $a_t$  for the next period. The Bellmann equation is usually not solvable analytically, but can be solved numerically via an iterative technique called successive approximation<sup>10</sup>. First we make an initial "guess"  $\nu_0(\cdot)$  for the value function, and define the next estimates iteratively:

$$\nu_{n+1}(s_t, \mathbf{a}_{t-1}) = \max_{a_t} \{pq(a_t, \mathbf{a}_{t-1}, \mathbf{W}) + z\nu_n(s_{t+1}, \mathbf{a}_t)\}$$

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<sup>8</sup> $I$  stands for increasing,  $D$  for decreasing

<sup>9</sup>The weights for the Ad-Stock have been determined by trying out all kind of combinations over a limited area, and pick the weights which produce the best fit in the final model.

<sup>10</sup>Note that this technique is computationally intensive, and the more past  $a_t$  are included in  $A_t$  the more complicated the procedure gets ("curse of dimensionality").

Under certain assumptions (which are fulfilled in Kamp & Kaiser [17])  $\nu_n$  will converge to  $\nu$  and the iteration can be stopped if the distance between  $\nu_n$  and  $\nu_{n+1}$  becomes small enough. Furthermore the derived policy function  $h$  will be single-valued, which means that there exists a single unique optimal policy.

The results for the New York milk market revealed that a steady 6-months cycle (with periods of zero advertising and periods with maximum advertising) would lead to the best result. The actual length of high and low advertising depend on the actual upper bound for monthly advertising levels. A quite similar result as with the continuous ADPULS model.

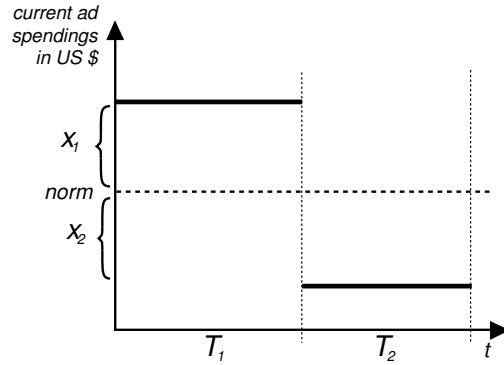


Figure 4.5: advertising policy modelled by Ambar Rao

## 4.7 Appendix: Rao's pulsing model

### Overview

The following should give an overview of the pulsing model introduced by Ambar G. Rao [21], chapter 5. One of the aims of this appendix will furthermore be, to provide a clean formal representation of the model itself and its parameters. Note, that in order to emphasize the functional dependencies between the inputs and the parameters, we use a slightly different notation here than Rao.

Firstly, Rao restricts the advertising policy of a company to the following special form: The company (with brand  $X$ ) starts out with a period of high advertising, which is then followed by a period of low advertising, whereas the extra spending ( $x_1$ ) and the saved spending ( $x_2$ ) even out over one cycle ( $T_1 + T_2$ ).

Secondly, Rao takes two different effects of advertising into account. These are changes in sales due to *switching* of consumers from one brand to another, and changes in sales due to a *change in consumption* by the consumer. Figure 4.7 displays the simplifying assumptions, which have been made by Rao for his model.

A consumer is said to be a  $X$  consumer, if his most preferred brand is  $X$ , resp.

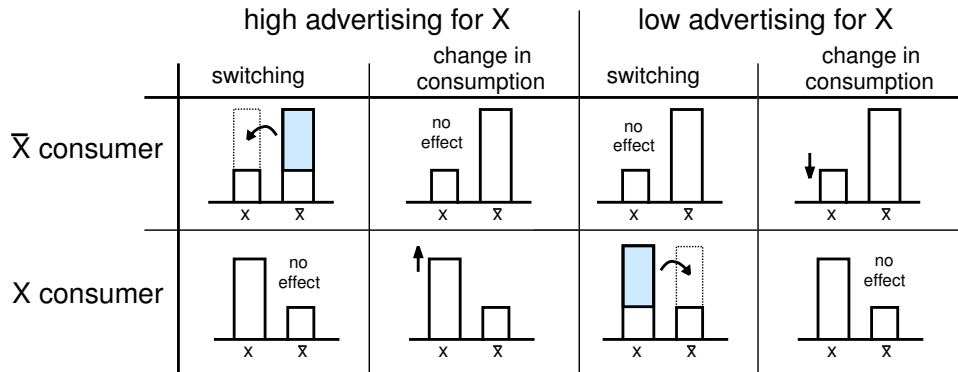


Figure 4.6: impact on the purchases of a single customer with regard to his preferred brand

a  $\bar{X}$  consumer if this is not the case<sup>11</sup>. The bars in the figure represent the brand shares for a *single* consumer, and also the different effects of advertising on these. E.g. Rao assumes that in a phase of high advertising for  $X$  switching can only occur for consumers, who do not already prefer brand  $X$ . This implies that there is no shifting of brands for  $X$  consumers during this phase (look at the lower left bar chart for this).

Furthermore we classify our market regarding to the loyalty level of consumers into several groups, and model their response function, and their response time separately. It is assumed that an effect to occur for a loyal consumer takes more time than for a nonloyal consumer. It is also assumed that the impact on sales of a brand due to switching is higher for loyal consumers, but lower due to a change in consumption.

**output**

$S$  (expected) change in sales per time unit due to pulsing for one cycle <sup>12</sup>  
of high and low advertising (measured in dollars)

<sup>11</sup>whereas Rao never defines the scope of time, over which the brand shares for a single consumer should be measured

<sup>12</sup>Rao tends to neglect the fact that we actually calculate an expected value

		impact on sales	
		switching	change in consumption
loyal consumer		large	small
	nonloyal consumer	small	large

Figure 4.7: impact on the purchases of a single customer with regard to his loyalty

**inputs**

- $x_1$  advertising spending per time unit above *norm* (nonnegative; in dollars);  
whereas the *norm* is defined as the advertising level at which sales remain constant
- $x_2$  advertising spending per time unit below *norm* (nonnegative; in dollars)
- $T_1$  length of time of high advertising
- $T_2$  length of time of low advertising

**parameters**

- $N_2$  number of  $X$  consumers at  $t = 0$   
i.e. consumers which have  $X$  as their most preferred brand
- $N_1$  number of  $\bar{X}$  consumers at  $t = 0$   
i.e. consumers which do not have  $X$  as their most bought brand
- $Z_i$  level of loyalty for market segment  $i$ ;  $i$  ranges from 1 to  $k$ ;  $Z_i < Z_{i+1}$
- $p_i$  proportion of consumers at time  $t = 0$  with loyalty level  $Z_i$
- $a_1, a_0, a_2$  parameters of the response function due to switching
- $a_1, a_3, a_4$  parameters of the response function due to change in consumption
- $\rho$  parameter for modelling reaction time of consumers

13 14

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<sup>13</sup>see the next section for a definition of the loyalty  $Z_i$

<sup>14</sup>It is assumed that the initial distribution of loyalty ( $p_i$ ) is equal for  $X$  and  $\bar{X}$  consumers; an assumption which is already clearly violated after the first cycle (i.e. at time  $T_1 + T_2$ ).

the pulsing model:

$$\begin{aligned} S(x_1, x_2, T_1, T_2) &= \frac{1}{T_1 + T_2} \left[ \int_0^{T_1} \Gamma(t, x_1) dt + \int_0^{T_2} \Lambda(t, x_2, x_1, T_1) dt \right] \\ x_1 T_1 - x_2 T_2 &= 0 \\ x_2 &\leq \text{norm} \end{aligned}$$

$\Gamma(\mathbf{t}, \mathbf{x}_1)$  is the (expected) additional sales level (in dollars) at time  $t$  (for  $t \leq T_1$ )

$$\begin{aligned} \Gamma(t, x_1) &= N_1 \sum_{i=1}^{\min[k, m_1(x_1)]} p_i b_i(x_1) \xi_i(t) \\ &+ N_2 \sum_{i=1}^{\min[k, m_1(x_1)]} p_i c_i(x_1) \xi_i(t) \end{aligned}$$

$\Lambda(\mathbf{t}, \mathbf{x}_2, \mathbf{x}_1, \mathbf{T}_1)$  is the (expected) loss in sales level (in dollars) at time  $T_1 + t$  (for  $t \leq T_2$ )<sup>15</sup>

$$\begin{aligned} -\Lambda(t, x_2, x_1, T_1) &= (N_2 + N_3(x_1, T_1)) \sum_{i=1}^{\min[k, m_2(x_2)]} q_i(x_1, T_1) b_i(x_2) \xi_i(t) \\ &+ (N_1 - N_3(x_1, T_1)) \sum_{i=1}^{\min[k, m_2(x_2)]} v'_i(x_1, T_1) c_i(x_2) \xi_i(t) \end{aligned}$$

$\xi_i(\mathbf{t})$  denotes the probability that a change has occurred for a consumer of loyalty  $Z_i$ . Rao argues that these probabilities can be modelled with a *gamma-distribution* (see Rao [21] p.64), with the parameter  $n$  of the gamma-distribution depending on the loyalty level. Note, that the same probability distribution is used for modelling the occurrence of switching *and* of change in consumption.

$$\xi_i(t) = \int_0^t \rho^i \frac{u^{i-1}}{(i-1)!} e^{-u\rho} du$$

---

<sup>15</sup>Rao omits the negative sign in this formula.

$\mathbf{b}_i(\mathbf{x})$  is the change in sales to a *single* consumer with loyalty  $Z_i$  due to switching, given that a switch occurs

$$b_i(x) = Z_i[a_0(x - a_1Z_i) + a_2(x - a_1Z_i)^{1/2}]$$

$\mathbf{c}_i(\mathbf{x})$  is the change in sales to a *single* consumer with loyalty  $Z_i$  due to change in consumption, given that a change occurs

$$c_i(x) = \frac{1}{Z_i}[a_3(x - a_1Z_i) + a_4(x - a_1Z_i)^{1/2}]$$

$\mathbf{m}_{1,2}$  are the maximum indices of a market segment, for which an effect on sales still takes place.

$$m_j(x_j) = \max i \text{ such that } x_j - a_1Z_i > 0, j = 1, 2$$

$\mathbf{N}_3(\mathbf{x}_1, \mathbf{T}_1)$  is the (expected) number of consumers who switch from  $\bar{X}$  to  $X$

$$N_3(x_1, T_1) = \sum_{i=1}^{\min[k, m_1(x_1)]} N_1 p_i \xi_i(T_1)$$

$\mathbf{q}_i(\mathbf{x}_1, \mathbf{T}_1)$  is the (expected) proportion of  $X$  consumers at time  $T_1$  with loyalty level  $Z_i$

$$q_i(x_1, T_1) = {}^{16} \begin{cases} \frac{N_2 p_i + N_1 p_i \xi_i(T_1)}{N_2 + N_3} & i \leq m_1 \\ \frac{N_2 p_i}{N_2 + N_3} & i > m_1 \end{cases}$$

$\mathbf{v}'_i(\mathbf{x}_1, \mathbf{T}_1)$  is the (expected) proportion of  $\bar{X}$  consumers at time  $T_1$  with loyalty level  $Z_i$

$$v'_i(x_1, T_1) = \begin{cases} \frac{N_1 p_i - N_1 p_i \xi_i(T_1)}{N_1 - N_3} & i \leq m_1 \\ \frac{N_1 p_i}{N_1 - N_3} & i > m_1 \end{cases}$$

## Clarifications

*Ad market segmentation:* In order to give an overview of the introduced variables regarding market size, and distribution of loyalty among its consumers, we provide the following table:

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<sup>16</sup>Rao falsely does not make a case differentiation here (see Rao [21] p.65)

at time $t = 0$		at time $t = T_1$	
	$X$	$\bar{X}$	
$Z_1$	$N_2 p_1$	$N_1 p_1$	$Z_1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$Z_k$	$N_2 p_k$	$N_1 p_k$	$Z_k$

a small market share we have a positive effect due to switching. This becomes comprehensible by considering that with a small market share we have a large pool of non- $X$  consumers, who might switch to  $X$  during the period of high advertising. As long as the market share does not grow too big, the effect of switching away from  $X$  during the period of low advertising will remain smaller (see Rao [21], figure 5.6 & 5.7). The effects of a change in consumption are exactly the opposite, i.e. with a low market share, we will have a negative impact on sales due to a pulsing policy.

Another (minor) result, which Rao concludes is, that the higher the amplitude of the pulsing ( $x_1$ ) is, the higher is the optimal  $T_2/T_1$  relation (see Rao [21], figure 5.9).

### **Limitations (resp. 'conceptual weaknesses')**

After building up this highly complex model with all its stated assumption and parameters, we will try to identify its benefits/consequences on our practical work.

The parameters (and even the number of parameters) of the model are, as Rao points out himself, practically impossible to estimate with an accurate error level for the resulting model. But by trying to estimate  $\Gamma(t, x_1)$  and  $\Lambda(t, x_2)$  directly by *the time paths of sales in periods of high advertising and low advertising*, as suggested in Rao [21] p.77, we would completely discard the fact, that (according to the model) we deal with different loyalty levels among our customers at different points of time. Therefore the whole effort of classifying the consumer market regarding loyalty and calculating their proportions would lead ad absurdum.

Though, one of the major benefits, according to Rao, is that we now have at least *a clue as to the shapes of these functions*. But these functions are basically nothing more than a linear transformation of the  $\xi_i(t)$ , which have been the modelled reaction times of the consumers, which furthermore have been argued to be gamma-distributions. But neither the type of distribution, nor the determination of the parameter  $n$  (which has actually been assumed to be the index of the loyalty level  $Z_n$ !) stand on a firm foundation.

And since we have no indication at to how many market segments ( $k$ ) we should build, we are actually able to model a wide range of monotonous function by linear combinations of the gamma-distribution. And therefore we have hardly any restrictions for the shapes of our functions.

So, after questioning the immediate benefits of Rao's model, we furthermore will try to point out a number of inherent limitations in the following paragraphs.

Most importantly we have to criticize the *limited time horizon*. We solely optimize for one cycle and do not, as the methods of Dynamic Programming strongly suggest, include the state of the system after the cycle into our considerations. We do not consider how many  $X$  consumers remain, or how they are distributed over the different loyalty categories, and therefore can not make any statements on the long-term optimality of our decisions.

Another drawback is the assumption of a '*lost memory*', whenever we switch from high to low advertising or vice versa. In this model past advertising completely loses its effect as soon as the advertising level is changed. Therefore it is not possible to build up a long-term brand value at all.

Rao's definition of *brand loyalty* perfectly fits in a row of several others, which solely provide an operational definition without any conceptual backing (see Jacoby & Chestnut [14]). Quote: "*the procedures measure BL*<sup>18</sup> *and BL is what the procedures measure*" ([14] p.73). Interestingly to note, that Rao's definition is not even among those 53(!) recognized definitions listed by Jacoby & Chestnut 8 years later.

But even the operational definition itself is problematic, since Rao does not provide any time frame for when the proportions of brands should be measured. Furthermore, as has been mentioned before, Rao does not indicate on how the discrete form of the measure looks like, and how many different levels of loyalty we actually can measure.

Rao's definition takes brand loyalty as a brand-independent measure for a single customer, which remains constant over time. This implies that advertising can have no effect on brand loyalty itself!

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<sup>18</sup>abbrev. for brand loyalty

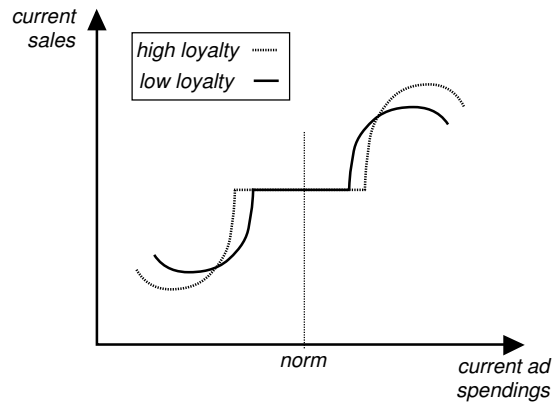


Figure 4.8: assumed shapes of the sales response function

To sum it up, advertising has no effect on brand loyalty, has no effect on the probability of a change to occur, and is lost whenever the advertising level is changed. A highly questionable concept.

Next drawback we identified is the discontinuous character of the effects of advertising. There is for example either a complete switch towards a brand for a single customer, or none at all. We do not model any kind of continuous transfers (neither continuous in time nor continuous in amount). See figure 4.7 for further insight.

We should further mention that no consideration of advertising activities of other market players has been included into this model.

Finally we have to question the stated advertising response functions. A graphical visualization of these reveals their arbitrary character. And since we already challenged Rao's definition of loyalty, we especially look mistrustful at the stated proportional relationship between this parameter and the response function.

## 4.8 Appendix: Simon

The optimization problem is of the following form:

$$\max_{A_{t+\tau}, \tau \geq 0} \Pi_t = \max_{A_{t+\tau}, \tau \geq 0} \sum_{\tau=0}^{\infty} [(p - C')q_{t+\tau} - A_{t+\tau}]z^\tau$$

Simon derives optimum levels for  $A_{t+\tau}$  by setting  $\frac{\partial \Pi_t}{\partial A_t} = 0$ , but does not consider  $\frac{\partial \Pi_t}{\partial A_{t+\tau}} = 0$  for  $\tau \geq 1$ .

For the simpler case of a symmetric response function (i.e.  $c = 0$ ) we will now perform the complete optimization, which has been omitted in the referred paper.

$$\begin{aligned} q_{t+\tau} &= \frac{a}{1-\lambda} + b \sum_{j=0}^{\infty} \lambda^j \ln A_{t+\tau-j} \\ \Pi_t &= \sum_{\tau=0}^{\infty} [(p - C') \frac{a}{1-\lambda} + (p - C')b \sum_{j=0}^{\infty} \lambda^j \ln A_{t+\tau-j} - A_{t+\tau}]z^\tau \\ \frac{\partial \Pi_t}{\partial A_{t+k}} &\stackrel{\underbrace{=}}{=} \sum_{\tau=k}^{\infty} [(p - C')b \lambda^{\tau-k} \frac{1}{A_{t+k}}]z^\tau - z^k = 0, \forall k \geq 0 \\ z^k &= z^k \frac{1}{A_{t+k}} b(p - C') \underbrace{\sum_{i=0}^{\infty} \lambda^i z^i}_{1/(1-\lambda z)}, \forall k \geq 0 \end{aligned}$$

Therefore the optimal policy is a constant spending of

$$A_{t+k}^* = b \frac{p - C'}{1 - \lambda z}, \forall k \geq 0.$$

# Chapter 5

## Game Theory

### 5.1 Motivation

As a motivating example for taking a game theoretical approach towards advertising we will reformulate the famous prisoner's dilemma for a duopolistic market.<sup>1</sup>

The following simple scenario constitutes the original prisoner's dilemma: Two criminals are kept in prison in separated cells while awaiting their trial at court. They are accused of having committed a serious crime together, but due to lack of evidence the indictment so far stands on a weak foundation. Therefore, if neither one of the criminals admits the crime, they can only be sentenced to one year in prison for some minor charges. Now in case one of them admits the crime and at the same time blames the other one of being the mastermind of the two of them, the criminal who cooperates with the police will go free, while the other one will face a 9-year sentence. In case that both prisoners are separately admitting the crime and are blaming each other, there is enough evidence for the judge to put both of them for 7 years in jail. If we now value each single year in prison with a 'negative utility' of one, then we derive the following symmetric 2x2 matrix as the strategic form for this game:

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<sup>1</sup>This reformulation was motivated by a comment in Feichtinger et al. (1994) [8] page 219. Other reformulations of the prisoner's dilemma can be also found in Mehlmann (2000) [18].

**Prisoner's Dilemma**

	B admits	B denies
A admits	7/7	0/9
A denies	9/0	1/1

It is easily deductable that the unique Nash equilibrium is attained with the strictly dominant strategy corresponding to the upper left corner. Independent of the decision of the other player it will always result in a lower negative utility for herself, if the person admits the crime, and therefore this is clearly the preferred strategy. But since we are dealing with a symmetric game, the same can be argued for the second player and therefore we end up in the situation where both players will admit the crime and are going to be imprisoned for the next seven years. The obvious dilemma is of course, that this scenario is not the optimal solution which could be attained for the players if both of them would deny the crime.

Despite its simplicity this game has been of immense significance for the field of game theory, and, if we believe the words of Alexander Mehlmann [18], has been able to rob researchers their sleep and in some cases even their sanity ever since its formulation.

If we now turn our focus to the competitive field of advertising, we can detect a quite similar scenario. There seems to be some common belief, respectively indications, that companies are persistently overadvertising, i.e. spending more than optimal on advertising. One of these indications is for example the low advertising elasticities which are commonly estimated in advertising sales response functions<sup>2</sup>. An indication for the cause of this phenomena might be found in the following Advertiser's dilemma.

Let's look at a duopolistic market<sup>3</sup> where each company has to make the one-shot decision whether to advertise or not. It is assumed that advertising has solely competitive effects, but does not influence overall market size. Therefore the two companies are competing for market-shares in a market of fixed size. Further

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<sup>2</sup>Aaker and Carman [1] establish for example that over 75% of the studies under review resulted in a short-term elasticity estimate of below 0.1.

<sup>3</sup>i.e. a market with two competing suppliers

we explicitly exclude the possibility of cooperative measurements (i.e. binding agreements) of the two companies. If both companies are advertising, then they will both have advertising costs without making any gain in sales. If neither one of them advertises, nothing will change and we will therefore state a utility of zero for this scenario. But in the case that just one of the companies advertises, while the other one does not, the company that advertises will gain significant gains in market share. If we make the decisive assumptions, that the additional profit due to the gained market share exceed the advertising costs ( $C_{ad} - P_{ms} < 0$ ), and that this profit is equivalent to the loss of the competitor due to his lost market share ( $C_{ms} = P_{ms}$ ), then we come up with a scenario which is equivalent to the prisoner's dilemma. In the following utility matrix we assumed  $C_{ms} = 9$  and  $C_{ad} = 7$ .

#### Advertiser's Dilemma<sup>4</sup>

	B advertises	B doesn't advertise
A advertises	$C_{ad} + 0 = \mathbf{7}$	$C_{ad} - P_{ms} = \mathbf{-2}$
A doesn't advertise	$0 + C_{ms} = \mathbf{9}$	$0 + 0 = \mathbf{0}$

The Nash equilibrium is again established in the upper left corner, since each company is trying to minimize its 'negative utility'. Therefore the outcome of this scenario will be that both companies will spend money on advertising without having any return on investment<sup>5</sup>.

This model is of course a highly simplifying model, but should at least provide enough motivation for taking a closer look at game theoretical tools and methods, since these might provide some insights in the dynamic interdependencies between marketing decisions within duopolistic or oligopolistic<sup>6</sup> markets.

<sup>5</sup>A quite similar assumption is established by Aaker and Carman [1] p.59: *"In the aggregate, when primary demand is not expandable, competitive behavior in oligopoly will cause all advertisers to counter the moves of a competitor, so that the level of advertising may end up at a level that appears excessive."*

<sup>6</sup>An oligopoly is a market with more than two, but generally less than eight suppliers.

## 5.2 Incorporating Competition

Several levels of sophistication of incorporating competition into market response models can be distinguished.

The most basic one would probably be to simply include the total industry advertising expenditures (respectively the ratio between own advertising to total advertising) as an additional variable into the model (see e.g. Rao (1970) §7 [21]). The expected relation is that a higher total advertising level diminishes the effects of a firm's advertising expenditures.

Next advanced step of incorporating competition would be to model reaction functions.

*"Reaction function models are an attempt to formulate a competitive situation by introducing a function that captures the behavior of the competitors in response to the action by the firm under consideration"*

[8] §7.4

In a paper by Hanssens (1980) [13] for example, the author provides a framework which explicitly models competitive reactions, and applies it in the following on data from the U.S. domestic air travel market. Three different types of reaction effects are detected: *Intrafirm* effects, which reflect the joint usage of several marketing instruments within a firm, *simple competitive* effects, i.e. competitors are reacting with the same marketing instrument, and third, so called *multiple competitive reactions*, i.e. competitors use different instruments to react.

The major drawback of these approaches so far is that competitors are only seen as passive reactors, whose reaction functions are known to the competitors, but not as optimizers on their own<sup>7</sup>. The methods to overcome this flaw can be found in the field of game theory, specifically in the field of differential games.

What follows is a short introduction to differential games, to the different optimality concepts, and to solution procedures. Later we will discuss specific

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<sup>7</sup>compare Sethi, "Dynamic Optimal Control Models in Advertising: A Survey", SIAM Review, 19, 4 (October 1977a), 685-725 for a similar statement

examples of competitive control models and close this chapter with an empirical study by Chintagunta and Vilcassim (1992) [2].

### 5.3 Differential Games

*"Differential Games are dynamic game models used to study systems that evolve in continuous time and where the system dynamics can be described by differential equations"*<sup>8</sup> [3]

In applications on marketing we usually deal with deterministic, non-cooperative, two-player games in continuous time of the following general form:

$$\begin{aligned} J_1(u_1(t), u_2(t)) &= \int_0^T e^{-r_1 t} F_1(x(t), u_1(t), u_2(t), t) dt + e^{-r_1 T} S_1(x(T)) \\ J_2(u_1(t), u_2(t)) &= \int_0^T e^{-r_2 t} F_2(x(t), u_1(t), u_2(t), t) dt + e^{-r_2 T} S_2(x(T)) \\ \dot{x}(t) &= f(x(t), u_1(t), u_2(t), t), \quad x(0) \text{ given} \\ u_1(t) &\in U_1(x(t), u_1(t), u_2(t), t), \quad \forall t \in [0, T] \\ u_2(t) &\in U_2(x(t), u_1(t), u_2(t), t), \quad \forall t \in [0, T] \end{aligned}$$

*Notation:*  $x \in X \subseteq \mathbb{R}^n$  is the state vector, which might represent market share, sales volume, advertising stock (=goodwill), etc of the competitors. Accordingly the set of differential equations are also known as state equations.  $u_i$  are the control variables of the  $i$ -th company, which might be pricing, advertising, etc. or a whole mix of these marketing instruments.  $J_i$  are the objective functionals, which consist of a terminal payoff part (i.e. salvage value of the state at time  $T$ ) and an integral payoff part.

The focus on duopolistic markets (i.e. two-player games) in marketing literature is probably rather a consequence of the increasing complexity which arises with oligopolistic models, than the lack of real-world scenarios. But as the field

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<sup>8</sup>There is also the related field of so-called *difference games*, where the dynamics are described by difference equations.

of differential games will advance, more papers on oligopolies are expected to be published.

Regarding the amount of information that the players use for determining their optimal control, we distinguish three levels<sup>9</sup>: The players can base their strategies on time alone, they can base it on time and the current state vector, or thirdly, they can consider the complete state trajectory which has evolved so far in the game. Within differential games we make the assumption that the necessary information of the previous history of a game is sufficiently represented in the state vector, and therefore we discard the third type (for now). If we allow the control function to be dependent on time *and* on the states, i.e.  $u_i(x(t), t)$ , we speak of a *Markovian strategy*, or also known as *closed-loop strategy*, whereas in the case of a solely time-dependent control, i.e.  $u_i(t)$ , of an *open-loop strategy*.

Determining the optimal strategy for a noncooperative game is generally not an easy task, and actually several concepts exist of how to obtain such a solution. The most-common concept is the so-called Nash equilibrium.

*”a Nash solution [...] is secure in the sense that no player can obtain a better outcome by unilaterally deviating from his Nash strategy as long as the other player plays his Nash strategy.”* (Jorgenson (1982) [15])

The downside of this concept is, that it is not guaranteed that a unique Nash equilibrium is obtained, therefore further criteria for a solution have to be defined in case of multiple equilibria. One possible criteria could be for example *subgame perfectness*<sup>10</sup>.

The focus on noncooperative games is due to the fact that companies generally act in their own interest. It is important to note, that this does not necessarily exclude collusion, since this strategy still might occur as the optimal outcome of a noncooperative game. A cooperative optimality concept is the one by Pareto, which would allow binding agreements between the players. Looking back at the prisoner’s dilemma, the cooperative solution would have been attained, if both criminals had denied the crime.

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<sup>9</sup>see Dockner et al. (2000) page 29f [3] for an excellent discussion on this issue

<sup>10</sup>see Dockner et al. (2000) page 24 [3]

We will now state of how to obtain Markovian Nash equilibria in differential games:

Let us define the current-value Hamiltonian functions as

$$\begin{aligned} H_1(x, u_1, \phi_2, \lambda_1, t) &= F_1(x, u_1, \phi_2, t) + \lambda_1 f(x, u_1, \phi_2, t) \\ H_2(x, \phi_1, u_2, \lambda_2, t) &= F_2(x, \phi_1, u_2, t) + \lambda_2 f(x, \phi_1, u_2, t) \end{aligned}$$

and the maximized Hamiltonian as

$$\begin{aligned} H_1^*(x, \phi_2, \lambda_1, t) &= \max_{u_1} H_1(x, u_1, \phi_2, \lambda_1, t) \\ H_2^*(x, \phi_1, \lambda_2, t) &= \max_{u_2} H_2(x, \phi_1, u_2, \lambda_2, t) \end{aligned}$$

Then the following conditions for a Markovian Nash equilibrium  $(\phi_1(x, t), \phi_2(x, t))$  can be derived via Pontryagin's maximum principle<sup>11</sup>:

(i) maximum conditions

$$\begin{aligned} H_1^*(x, \phi_2, \lambda_1, t) &= H_1(x, \phi_1, \phi_2, \lambda_1, t), \forall t \in [0, T] \\ H_2^*(x, \phi_1, \lambda_2, t) &= H_2(x, \phi_1, \phi_2, \lambda_2, t), \forall t \in [0, T] \end{aligned}$$

(ii) adjoint equations

$$\begin{aligned} \dot{\lambda}_1 &= r_1 \lambda_1 - \frac{\partial H_1^*}{\partial x}(x, \phi_2, \lambda_1, t), \quad \lambda_1(T) = \frac{\partial S_1}{\partial x}(x(T)) \\ \dot{\lambda}_2 &= r_2 \lambda_2 - \frac{\partial H_2^*}{\partial x}(x, \phi_1, \lambda_2, t), \quad \lambda_2(T) = \frac{\partial S_2}{\partial x}(x(T)) \end{aligned}$$

The state equations and the adjoint equations together build a *two-point boundary value problem with coupled partial differential equations*, a problem which is generally hard to solve (not just analytically, but also numerically). But if we are limiting ourselves to open-loop controls, we have  $\frac{\partial \phi}{\partial x} = 0$  and therefore can derive *ordinary* differential equations, which are far easier to handle.

This fact might also be the reason for the popularity of open-loop differential games in marketing. Nevertheless their underlying assumptions are rather

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<sup>11</sup>see Dockner et al. (2000) Theorem 4.2 for a complete and wider formulation [3]

questionable because optimizers are disregarding available information in their decision-making process (i.e. the state vector  $x$ ). Further, the empirical study by Chintagunta and Vilcassim (1992) [2], which is discussed later, also indicates that closed-loop strategies provide a better fit to real-world data than open-loop ones do.

## 5.4 Competitive Control Models

In the following we will briefly introduce three models, whereas the first one will be a simple generalization of the Vidale-Wolfe model, and the other two are applications of the Lanchester Model of combat on duopolistic markets. In contrast to the control models analyzed in previous chapters we explicitly include the effects of market-share and advertising expenditures of the competitors, and have all of them act as individual optimizers on their own.

### 5.4.1 Vidale-Wolfe generalization

<sup>12</sup>Let  $x_i$  denote sales volumes and  $u_i$  advertising rates, and have  $M$  refer to the overall market potential, then the dynamics are described by

$$\begin{aligned}\dot{x}_1 &= b_1 u_1 \left(1 - \frac{x_1 + x_2}{M}\right) - a_1 x_1 \\ \dot{x}_2 &= b_2 u_2 \left(1 - \frac{x_1 + x_2}{M}\right) - a_2 x_2 \\ J_i &= \int_0^T (q_i x_i - u_i^2) dt + w_i \frac{x_i(T)}{x_1(T) + x_2(T)}, \quad i = 1, 2\end{aligned}$$

From the objective functional it can be seen that we model a linear revenue function, but decreasing returns on advertising by assuming quadratic costs.

Note, that within this model competitors advertising expenditures only influence the dynamics indirectly through their resulting sales volume. The higher the competitors sales are, the less market potential is available, and the less effective the advertising expenditures will be. Note that different response rates  $b_i$  and

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<sup>12</sup>see Jorgenson [15] p.349

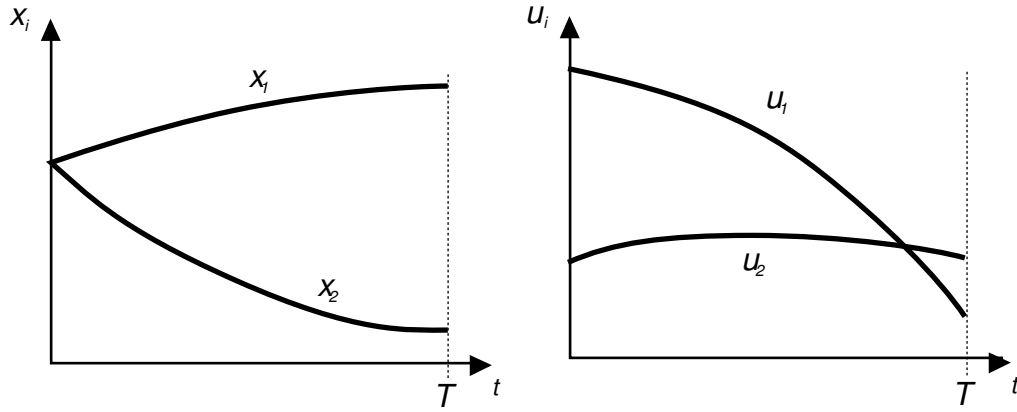


Figure 5.1: Optimal sales path for  $a_1 \ll a_2$

decay rates  $a_i$  are modelled for the two companies, and therefore the obtained optimal controls are likely to differ in dependency of these coefficients and their relation to each other.

The derived open-loop Nash controls imply that the company with the lower decay rate  $a_i$  should spend more on advertising, and will be able to achieve the higher sales volume than its competitor (see Figure 5.1).

Another important factor for the optimal control will be the duration  $T$ , and the relation between terminal payoffs and integral payoffs. The observed behaviour is, that in case that more weight is assigned to the salvage value, we will expect an increasing control towards time  $T$ , and vice versa.

### 5.4.2 A Lanchester-type model by Case

<sup>13</sup>Here we denote the market shares with  $x_i$  (i.e.  $x_1 + x_2 = 1$ ), and set the state equations to the following form:

$$\begin{aligned} \dot{x}_1 &= u_1(1 - x_1) - au_2x_1 \\ \dot{x}_2 &= -\dot{x}_1 \\ J_i &= \int_0^\infty e^{-rt}(q_ix_i - u_i^2)dt, \quad i = 1, 2 \end{aligned}$$

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<sup>13</sup>see Jorgenson [15] p.353

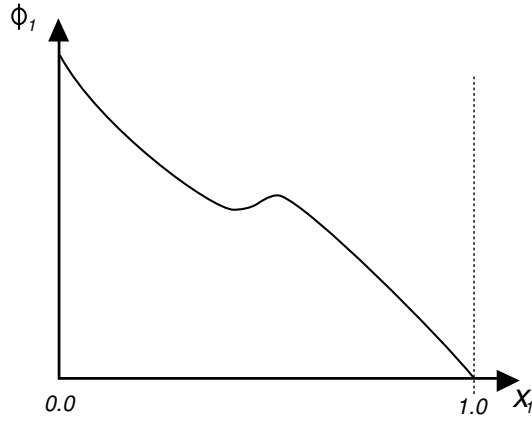


Figure 5.2: Markovian Nash equilibrium of firm 1

In this model we have an unlimited time horizon and a direct influence of the competitors advertising on the decay rate. The more the competitor advertises, the faster the customers will switch towards the competitor.

Case derives a Markovian Nash equilibrium for  $q_1 = q_2$  and  $a = 1$ , which turned out to be only state-dependent, and not time-dependent. The optimal strategy therefore just depends on the current market share, whereas the higher the market share, the lower the optimal advertising level is (see Figure 5.2).

### 5.4.3 A Modification of the Case Game by G. Sorger

<sup>14</sup>Let  $x_i$  denote the market share again, and model the dynamics as followed:

$$\begin{aligned} \dot{x}_1 &= u_1\sqrt{1-x_1} - u_2\sqrt{x_1} \\ \dot{x}_2 &= -\dot{x}_1 \\ J_i &= \int_0^T e^{-rt}(q_i x_i - (c_i/2)u_i^2)dt + e^{-rT} S_i x_i(T), \quad i = 1, 2 \end{aligned}$$

Sorger shows that these state equations are basically the Lanchester dynamics together with an extra term which takes excessive advertising (i.e. the difference

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<sup>14</sup>see Dockner et al. [3] p.286ff

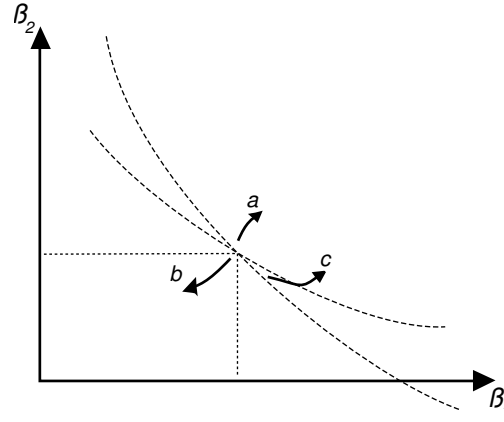


Figure 5.3: phase diagram

between  $u_1$  and  $u_2$ ) explicitly into account, and weights this factor with the social interactions between the two customer groups (i.e.  $x_1(1 - x_1)$ ).

The optimal closed-loop control is of the form

$$\phi_i(x_i, t) = \frac{\beta_i(t)}{c_i} \sqrt{1 - x_i}$$

whereas  $\beta_i(t)$  is the solution of

$$\dot{\beta}_i(t) = r_i \beta_i(t) - q_i + \frac{\beta_i(t)^2}{2c_i} + \frac{\beta_i(t)\beta_j(t)}{c_j}, \quad \beta_i(T) = S_i$$

Similar to the Case Game this implies that firms should choose to spend more on advertising the smaller their market share is and vice versa. In this particular model, Sorger further derives that an increase of  $2p\%$  of the rival's market share, should lead to a  $p\%$  increase of a firm's advertising effort.

By analyzing the phase diagram of this system (see Figure 5.3) we can detect a single unstable node, from which the optimal paths diverge. The dashed curves represent the isoclines  $\dot{\beta}_1 = 0$  and  $\dot{\beta}_2 = 0$ . Depending on the salvage values of the market shares at time  $T$  (i.e.  $S_1$ , resp.  $S_2$ ) three types of paths can be distinguished. If both  $S_i$  are "large" (relative to the steady state point), it is optimal to start out with a low advertising budget, and increase it steadily until time  $T$  (path  $a$ ). In the case of "small"  $S_i$  we obtain the opposite behavior as

optimal, i.e. decreasing advertising budget over time (path  $b$ ). In the case of  $S_1$  large, and  $S_2$  small, the optimal trajectory implies a steadily increasing  $u_1$ , whereas  $u_2$  should initially decrease, and hereupon increase again (path  $c$ ).

## 5.5 Empirical Study by Chintagunta and Vilcassim

In their article "An Empirical Investigation of Advertising Strategies in a dynamic Duopoly" [2] the two authors use a Lanchester-type model ( $\dot{M} = k_1\sqrt{a_1}(1-M) - k_2\sqrt{a_2}M$ ) to derive the optimal advertising spending of Pepsi and Coca-Cola between 1968 and 1981. They do so by first estimating the parameters  $k_1$  and  $k_2$  with the available data from this time period by using ordinary least square estimators, and then use methods of differential games to calculate numerically open-loop and closed-loop Nash equilibrium strategies for both market players. The assumed objective functionals are of the form

$$\max_{a_i} \pi_i = \int_0^{\infty} e^{-\rho t} (g_i M_i Q - a_i) dt, \quad i = 1, 2,$$

whereas  $Q$  denotes the overall sales volume of the market, and  $g_i$  is the profit margin (advertising costs excluded) of firm  $i$ .

It turned out, that the derived closed-loop strategies differed less from the actual advertising expenditures than the open-loop ones, which might imply that the latter one are able to provide a better fit to real-world scenarios. A result which makes intuitively sense, since the competitors in that case would not discard valuable information (i.e. the current market share) for their decision process. Further it is interesting to note that Pepsi seemed to operate closer to the "optimal solution" than its competitor.

## 5.6 Future Developments

A number of limitations currently exist in respect to differential games, which will hopefully be addressed by researchers in the near future:

- The assumption of *complete information* about the preference functions (i.e. cost structure, revenue function, etc.) of the competitors is usually not given in real-world scenarios.
- The introduced techniques further assume *rational behavior* of the involved players, whereas rationality here means that all players are using the *same* framework and the *same* underlying model and parameters to make their decisions.
- Due to difficulties of solving partial differential equations, established Markovian Nash equilibria are still rare in literature.
- The complete history of the competitors actions are only represented in the current state vector. This implies that behavior patterns of the market are difficult to model and therefore important information is discarded for the decision process.

Despite these existing limitations we are expecting an increasing number of applications of differential games in advertising. This seems inevitable in order to cope with the strong interdependencies between the marketing activities of the competitors within a duopolistic, respectively oligopolistic markets properly.

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